THE CAPM RELATION FOR INEFFICIENT PORTFOLIOS

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Abstract. Following empirical evidence that—contrary to CAPM predictions—found little relation between expected rates of return and betas, the relation has been investigated extensively. Roll (1977), Roll and Ross (1994) (RR) Kandel and Stambaugh (1995), and Jagannathan and Wang (1996) are seminal works. In this context, within a Markowitz world (finite number of nonredundant risky securities with finite first two moments), we generally and simply write the theoretical CAPM relation for inefficient (non-frontier) portfolios (CAPMI). We demonstrate that the CAPMI is a well-specified alternative for the widely implemented misspecified CAPM for use with inefficient portfolios. We identify three sources for this misspecification: *i*) the omission of an addend in the pricing relation, *ii*) the use of an incorrect risk premiums/beta coefficients (due to of the existence of infinitely many "zero beta" portfolios at all expected returns), and *iii*) the use of unadjusted betas. We suggest the use of incomplete information equilibria to overcome unobservability of moments of returns. Our results are robust to regressions that produce positive explanatory beta power, including extensions such as multiperiod, multifactor, and the conditioning on time and various attributes.

JEL Codes: G10, G12
Key Words: CAPM, beta, expected returns, incomplete information, zero relation

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1 Introduction

The simple and intellectually satisfying classical CAPM has been a main paradigm in finance. Thus, it was disconcerting to many believers when it appeared that empirical evidence offered little support to a CAPM basic prediction. Fama and French (1992), for example, found little relation between expected returns[1] and betas. Subsequently, this relation has been investigated extensively. For example, the seminal works of Roll and Ross (1994) (RR) and Kandel and Stambaugh (1995) (KS) argued that the problem is not in the model but in our inability to identify efficient proxy portfolios.

Careful reading, however, of "Roll's Critique," Roll (1977), would have forewarned us of misspecification while using inefficient proxies with the traditional CAPM. Researchers, however, largely ignored this point, perhaps because of Roll's Critique other seminal contributions.[2]

A Markowitz world (a finite set of nonredundant risky securities with finite first two moments) that has no further (equilibrium) assumptions induces an exact affine relation between expected returns and betas.[3] Quantitatively, this relation is identical to a classical CAPM relation, so we call this relation a classical CAPM type relation and denote it briefly CAPM. We use the word *type* to differentiate from a CAPM relation that arises in general equilibrium under extensive assumptions. We

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[1] Everywhere in the paper we use "expected returns" to briefly say "expected rates of return."

[2] Following Merton (1972) mathematical development of the portfolio frontier, Roll (1977) first emphasized that all, and only, mean-variance frontier portfolios induce a CAPM, thus the only CAPM testable implication is the efficiency of the index. Roll also emphasized that the GMVP (global minimum variance portfolio) is an exception, and that it induces beta equals to one on all assets.

[3] See, for example, Feldman and Reisman (2003) for a simple construction.

say, and explain why below, that the CAPM (and the equilibrium CAPM) is well defined for all reference portfolios excluding those with expected returns equal to that of the global minimum variance portfolio (GMVP).

In this context, we first develop a general and simple method to write the theoretical CAPM in terms of inefficient portfolios (CAPMI). We use the term "inefficient" portfolios to imply "non-frontier" portfolios, noting that the CAPM relation does hold for frontier portfolios on the negatively sloping part of the frontier. The CAPMI is more general than the CAPM, which is included in it. The CAPMI degenerates to the CAPM only in the special case of proxies that are on the portfolio frontier, and as a result one of the two beta addends of the CAPMI vanishes. The CAPMI facilitates a quick, simple, and clear demonstration of the additional results that we state below.

Second, we show that a theoretical zero relation between expected returns and betas (a zero coefficient of the betas in the CAPM restriction) may occur where the CAPM is not well defined. It occurs, however, only under a degenerate indeterminate case that non-uniquely allows a theoretical zero relation as one possible relation out of infinitely many non-zero possible ones. This occurs where

- the reference portfolio is in a degenerate cone in the mean-variance space, at the line where expected returns are equal to those of the GMVP
- all securities have betas equal to 1 and the same expected return
- there is no zero beta portfolio

On the other hand, where a CAPM is well defined, we very simply demonstrate that, as Roll (1980) showed, "Every nonefficient index possesses zero-beta portfolios at all levels of expected returns." [Roll (1980), p. 1011]. In particular, for any inefficient proxy there is at least one and could be infinitely many zero beta portfolios of the same expected return, which, in turn, implies that for any inefficient portfolio proxy there is at least one portfolio and could be infinitely many portfolios that induce zero relations. We provide a numerical example of a zero relation case with both exogenously given and endogenously constructed zero relations. Consequently, a zero relation could be empirically detected.

Third, our analysis emphasizes an essential implication: where the CAPM is well defined and where market portfolio proxies are inefficient, CAPM regressions are essentially misspecified because of three sources of misspecification. The first source of misspecification arises because the use of the CAPM for inefficient portfolios inappropriately and incorrectly ignores a non-zero addend in the restriction. The second source of misspecification arises from the, above mentioned, existence of infinitely many "zero beta" portfolios, and at all expected returns, *for any* inefficient market portfolio proxy. Thus, the identification of a correct "market risk premium," "excess return," or beta coefficient, is extremely unlikely. On the other hand, the identification of "zero relations" that induce a zero R^2 becomes possible. The third source of misspecification arises from the use of unadjusted betas, while adjusting the betas is required for inefficient proxies.

This misspecification is, of course, robust with respect to the explanatory power of the betas. Also subject to the misspecification are CAPM regressions that use different procedures from Fama and French's (1992) and that produce positive beta explanatory power. The misspecification is also robust to various extensions, such as multiperiod, multifactor, and the conditioning on time and various attributes. This CAPMI implication might be particularly beneficial as it is not clear that the RR KS, and Jagannathan and Wang (1996) essential implication—that CAPM regression with inefficient proxies are meaningless—has been sufficiently internalized.

Fourth, we suggest that applications/tests that use inefficient proxies should use our well-specified CAPMI rather than the misspecified CAPM for inefficient proxies.

Finally, because the real-world unobservability of moments of returns (a cause of the use of inefficient proxies) impairs the usefulness of the CAPMI, we suggest the implementation and testing of incomplete information equilibria models developed to handle unobservable moments, as demonstrated in Feldman (2005), for example.

For a simple construction of the CAPMI we use an orthogonal decomposition similar to the one in Jagannathan's (1996) finite number of securities version of Hansen and Richard's (1987) conditioning information model.

Given a finite number of nonredundant risky securities with distributions of rates of return that have finite means and variances, the Sharpe-Lintner-Mossin-Black[4] CAPM specifies an affine relation between security expected returns and betas. This relation holds for any portfolio frontier portfolio[5] (henceforth *frontier portfolio*), other than the GMVP. The coefficient of the beta in this affine relation is the expected return on the frontier portfolio in excess of the expected return of a portfolio that is uncorrelated with it (a zero beta portfolio). This excess expected return (for a frontier portfolio) cannot be zero.[6] We exclude the GMVP because a zero beta portfolio does not exist there, and the limit of the zero beta rate approaching the GMVP is infinite.[7] Thus, we say that the CAPM is well defined with respect to any frontier portfolio except for the GMVP.

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[4] Sharpe (1964), Lintner (1965), Mossin (1966), Black (1972).

[5] The portfolio frontier is the locus of minimum variance portfolios of risky assets for all expected returns.

[6] Roll (1977). See also Huang and Litzenberger (1988), Equation (3.14.2), which follows Merton (1972).

[7] The GMVP induces a beta of one on all securities. Geometrically, on a mean standard deviation Cartesian coordinates, the tangent to the GMVP is parallel to the expected return axis [Roll (1977)].

Roll (1980) demonstrated that there is a theoretical zero relation between expected returns and betas for every inefficient portfolio, where the CAPM is well defined. We provide intuition and simple construction of this result. Consider the hyperbola that an inefficient proxy spans with the GMVP and also the (degenerate) hyperbola it spans with the frontier portfolio of the same expected return. We demonstrate below that each of these hyperbolas includes a zero beta portfolio to the inefficient proxy and that these two zero beta portfolios are of different expected returns.

Now, all infinitely many portfolios, expanding to all expected returns, on the hyperbola spanned by these two zero beta portfolios are also zero beta with respect to the inefficient proxy. We call such a hyperbola a "zero beta hyperbola." In addition, any zero beta portfolio not on this hyperbola generates infinitely many additional portfolios that are zero beta with respect to the inefficient proxy. There are vast regions where infinitely many such portfolios may exist, and we give a numerical example for such a case. Thus, we have at least one and possibly infinitely many zero beta hyperbolas and on each such hyperbola infinitely many zero beta portfolios.

Because any non-degenerate zero beta hyperbola expands to all expected returns (as is the case for any non-degenerate hyperbola), it includes a portfolio with expected return equal to that of the inefficient proxy. Moreover, there are infinitely many portfolios on each zero beta hyperbola that induce incorrect "excess expected return" values (risk premiums/beta coefficients) in the CAPM relation. Thus, any inefficient proxy induces incorrect pricing due to incorrect excess expected return premia with respect to infinitely many portfolios. When these excess expected returns are zero—that is, the expected returns of the zero beta portfolios are equal to that of the inefficient proxy—they induce zero relations. Of course, each of these infinitely many portfolios, whether inducing an incorrect excess expected return value or a zero relation, induces a pricing error.

Therefore, where the CAPM is well defined, using it with inefficient proxies gives rise to three sources of misspecification. The first is ignoring a non-zero addend in the relation, the second is using an incorrect excess expected return value, and the third is using an incorrect value for beta. Recapping, the reason for the first and the third sources of misspecification is the need to correct the inefficient proxy “coordinates” to efficient ones on which the CAPM is defined, the reason for the second is that inefficient proxies have infinitely many zero beta portfolios and of all expected returns.

Where the CAPM is not well defined, there is a special case of degenerate indeterminacy that (non-uniquely) allows a theoretical zero relation. This case requires, however, that all securities have the same expected return. The explanation is as follows. If all securities have the same expected return,

(a) the portfolio frontier degenerates to one point that also becomes the efficient frontier,

(b) the proxy portfolio must be of the same expected return as the GMVP,

(c) all securities’ betas are equal to one, and

(d) a zero beta portfolio does not exist. Then, for any constant, there are infinitely many pairs of weights that average the constant and 1 (where 1 stands for any security’s beta), such that the average is equal to the securities’ expected return. In particular, there is a constant (the expected return of the market securities) that induces a theoretical zero relation (a zero weight on the beta). Thus, an implication of a Markowitz world is that a theoretical zero relation exists only if all securities have the same expected return.

While the analysis in this paper is done in a single-period mean-variance framework, its implications apply to multiperiod, multifactor models. This is because we can see the single period mean-variance model here as a “freeze frame” picture of a dynamic equilibrium where, because of the tradeoff between time and space, only the instantaneous mean and instantaneous variance of returns are relevant until the decision is next revised in the next time instant.

Roll (1977), RR, KS, and Jagannathan and Wang (1996), perhaps the seminal articles in this context, elaborately discuss the relation between expected returns and betas and its implications for regression estimates [see also the report of some of their results in Bodie, Kane, and Marcus (2005), Section 13.1, page 420]. We complement their results by specifying the CAPMI and demonstrating properties of the theoretical relation; see RR, KS, and Jagannathan and Wang (1996) for detailed perspectives and references. In a different context, Ferguson and Shockley (2003) examine the implications of omitting “debt” from the market portfolio and show that equity only proxies induce understated betas. We, indeed, obtain similar property for all inefficient proxies. Section 2 demonstrates the results, Section 3 discusses implications, and Section 4 concludes.

2 The CAPM Relation

Below, we introduce the model—a Markowitz world—and develop the analytical results.

2.1 Markowitz World and CAPM

In this section, we present the economy and write a CAPM using the following notational conventions: constants and variables are typed in italic (slanted) font, operators and functions in straight font, and vectors and matrices in boldface (dark) straight font.

In a market with *N* risky securities, let **R** be an $N \times 1$ vector of rates of return of the securities, R_i, $i = 1,...,N$, and $N > 2$. We do not specify the probability distributions of the rates of return. Rather, we assume means and variances that are real finite numbers and a positive definite covariance matrix, **V**, which implies that there are no redundant securities.[8] This non-redundancy, in turn, implies that there are at least two securities with distinct expected returns and a non-frontier security. We call the vector of security expected returns **E**, the expectation operator $\mathrm{E}(\cdot)$, the covariance $\sigma_{ij}, \forall R_i, R_j$, the variance $\sigma_{ii} \triangleq \sigma_i^2, \forall R_i$, and the standard deviation $\sigma_i \triangleq +\sqrt{\sigma_i^2}, \forall i$.

Let some portfolio, say **a**, of the *N* market securities, be an $N \times 1$ vector of real numbers, with components a_i, $i = 1,...,N$, where a_i is the "weight" of security *i* in the portfolio and, unless otherwise noted, $\mathbf{1}^{\mathrm{T}}\mathbf{a} = 1$, where $\mathbf{1}$ is an $N \times 1$ vector of ones and the superscript T denotes the transpose operator. Let z be a zero beta operator, i.e., z**a** is portfolio **a**'s zero beta frontier portfolio; thus, by definition, $\sigma_{\mathrm{z}\mathbf{aa}} = \sigma_{\mathbf{a}\mathrm{z}\mathbf{a}} = 0$. We will call some portfolio that is uncorrelated with **a,** thus having a zero beta with respect to **a**, *z***a**.[9] We call this world a Markowitz world.[10]

Let **q** be some frontier portfolio other than the GMVP. Portfolio **q** stands for a frontier index or reference portfolio. Then, we can write a Sharpe-Lintner-Mossin-Black (zero beta) CAPM for **q**:

[8] We define a redundant security as one whose return can be constructed by combining other securities.
[9] For simplicity of notation, and consistently with our notation convention, we use the operator z (z in straight font) on some portfolio **a**, z**a** to delineate the frontier portfolio that is uncorrelated with **a**, and we call some portfolio (not necessarily a frontier portfolio) that is uncorrelated with **a**, *z***a** [z is in slanted font (italics)]. The visual distinction between z**a** and *z***a** is subtle, but in context, there is little ambiguity and the introduction of additional notation is unnecessary.
[10] Markowitz (1952), for example. See also Roy (1952).

$$\mathbf{E} = \mathrm{E}(R_{\mathbf{zq}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathbf{zq}})]\frac{\mathbf{Vq}}{\sigma_{\mathbf{q}}^2}.^{11} \tag{1}$$

2.2 CAPMI

In this section, we write a CAPMI in terms of any portfolio—efficient or inefficient—excluding those with an expected return equal to that of the GMVP where the CAPM is not well defined. The previous section's CAPM is, thus, a special case of this section's CAPMI.

Let **p** be a portfolio with $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{q}})$ and $\sigma_{\mathbf{p}} > \sigma_{\mathbf{q}}$. Portfolio **p** stands for an inefficient portfolio that serves as a proxy to **q**. In a mean-standard deviation Cartesian coordinate system where the mean is on the vertical axis, **q** lies on the frontier and **p** lies inside the frontier to the right of **q**.[12]

We project $R_{\mathbf{p}}$ on $R_{\mathbf{q}}$, decomposing it into $R_{\mathbf{q}}$ and a residual return $R_{\mathbf{e}}$:

$$R_{\mathbf{p}} = R_{\mathbf{q}} + R_{\mathbf{e}}, \tag{2}$$

implying

$$\mathbf{p} = \mathbf{q} + \mathbf{e}, \tag{3}$$

where $\mathrm{E}(R_{\mathbf{e}}) = 0$, $\sigma_{\mathbf{qe}} = 0$, $\mathbf{1}^{\mathrm{T}}\mathbf{e} = 0$, $\sigma_{\mathbf{pq}} = \sigma_{\mathbf{q}}^2$, $\sigma_{\mathbf{pe}} = \sigma_{\mathbf{e}}^2$, $\sigma_{\mathbf{e}} > 0$, and **e** is the weights vector of $R_{\mathbf{e}}$. The orthogonal decomposition in Equations (2) and (3) is similar to those in Hansen and Richard (1987), (see, there, Equation 3.7, p. 596), and Jagannathan (1996), (see, there, Equation 1, p. 3).

We will now demonstrate why Equation (2) and the six following properties hold. Equation (2) and the first two properties hold because we can project any portfolio **p** on any portfolio **q** such that $R_{\mathbf{p}} = c + bR_{\mathbf{q}} + R_{\mathbf{e}}$, where *c* and *b* are constants,

[11] For a simple construction, see Feldman and Reisman (2003); for a geometric approach, see Bick (2004); and for a frontier expansion, see Ukhov (2005).
[12] For an examination of inefficient portfolios, see Diacogiannis (1999).

$\mathrm{E}(R_{\mathbf{e}})=0$, and $\sigma_{\mathbf{qe}}=0$. We achieve this if we choose $b=\frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}$, and $c=\mathrm{E}(R_{\mathbf{p}})-\frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}\mathrm{E}(R_{\mathbf{q}})$.[13] The choice that $\mathrm{E}(R_{\mathbf{p}})=\mathrm{E}(R_{\mathbf{q}})$ implies that $c=0$, $b=1$, and, by left multiplying Equation (3) by $\mathbf{1}^{\mathrm{T}}$, that $\mathbf{1}^{\mathrm{T}}\mathbf{e}=0$.[14] Equation (3) implies that $\sigma_{\mathbf{pq}}=\sigma_{(\mathbf{q+e})\mathbf{q}}=\sigma_{\mathbf{q}}^2+\sigma_{\mathbf{qe}}$ and that $\sigma_{\mathbf{pe}}=\sigma_{(\mathbf{q+e})\mathbf{e}}=\sigma_{\mathbf{e}}^2+\sigma_{\mathbf{qe}}$. Together with $\sigma_{\mathbf{qe}}=0$, we have $\sigma_{\mathbf{pq}}=\sigma_{\mathbf{q}}^2+\sigma_{\mathbf{qe}}=\sigma_{\mathbf{q}}^2$, and $\sigma_{\mathbf{pe}}=\sigma_{\mathbf{e}}^2+\sigma_{\mathbf{qe}}=\sigma_{\mathbf{e}}^2$. Finally, because $\sigma_{\mathbf{qe}}=0$ Equation (2) implies that $\sigma_{\mathbf{p}}^2=\sigma_{\mathbf{q+e}}^2=\sigma_{\mathbf{q}}^2+\sigma_{\mathbf{e}}^2+2\sigma_{\mathbf{qe}}=\sigma_{\mathbf{q}}^2+\sigma_{\mathbf{e}}^2$. Thus, the property $\sigma_{\mathbf{p}}>\sigma_{\mathbf{q}}$ implies that $\sigma_{\mathbf{e}}>0$.

Equation (2)'s projection is similar to regressing $R_{\mathbf{p}}$ on $R_{\mathbf{q}}$. Equivalently, this is a market model presentation of $R_{\mathbf{p}}$, developed in Sharpe (1963).

Substituting $\mathbf{p}=\mathbf{q}+\mathbf{e}$ into Equation (1) yields

$$\mathbf{E}=\mathrm{E}(R_{z\mathbf{q}})\mathbf{1}+[\mathrm{E}(R_{\mathbf{q}})-\mathrm{E}(R_{z\mathbf{q}})]\frac{\mathbf{V}(\mathbf{p}-\mathbf{e})}{\sigma_{\mathbf{q}}^2}. \tag{4}$$

When we rearrange and define $\boldsymbol{\beta}_{\mathbf{p}}\triangleq\frac{\mathbf{Vp}}{\sigma_{\mathbf{p}}^2}$, and $\boldsymbol{\beta}_{\mathbf{e}}\triangleq-\frac{\mathbf{Ve}}{\sigma_{\mathbf{e}}^2}$ as vectors of market security betas with respect to portfolios **p** and **e** respectively, Equation (4) becomes

$$\mathbf{E}=\mathrm{E}(R_{z\mathbf{q}})\mathbf{1}+[\mathrm{E}(R_{\mathbf{q}})-\mathrm{E}(R_{z\mathbf{q}})]\frac{\sigma_{\mathbf{p}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{p}}+[\mathrm{E}(R_{\mathbf{q}})-\mathrm{E}(R_{z\mathbf{q}})]\frac{\sigma_{\mathbf{e}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{e}}. \tag{5}$$

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[13] The (orthogonal) decomposition $R_{\mathbf{p}}=c+bR_{\mathbf{q}}+R_{\mathbf{e}}$, $\sigma_{\mathbf{qe}}=0$ implies $\sigma_{\mathbf{qe}}\triangleq\mathrm{COV}(R_{\mathbf{q}},R_{\mathbf{e}})=\mathrm{COV}(R_{\mathbf{q}},R_{\mathbf{p}}-bR_{\mathbf{q}})=\sigma_{\mathbf{pq}}-b\sigma_{\mathbf{q}}^2=0$, which, in turn, implies $b=\frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}$. If we choose *b* as implied, and, in addition, choose *c* to equal $c=\mathrm{E}(R_{\mathbf{p}})-b\,\mathrm{E}(R_{\mathbf{q}})=\mathrm{E}(R_{\mathbf{p}})-\frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}\mathrm{E}(R_{\mathbf{q}})$, we also have $\mathrm{E}(R_{\mathbf{e}})=0$ and accomplish the decomposition.

[14] With $\mathrm{E}(R_{\mathbf{e}})=0$ and $\mathbf{1}^{\mathrm{T}}\mathbf{e}=\mathbf{0}$, **e** is an arbitrage portfolio.

Equation (1) implies that portfolios with expected returns equal to that of z**q** are uncorrelated with **q**.[15] In addition, zz**q** is **q**. Thus, all portfolios with the same mean as **q** are uncorrelated with z**q**. Therefore, because we have $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{q}})$, we also have $z\mathbf{q} = z\mathbf{p}$. That is, the frontier portfolio that is zero beta with respect to **q** is zero beta with respect to all portfolios of the same expected return equal to that of **q**, including, in particular, **p**. Thus, $\mathrm{E}(R_{z\mathbf{p}}) = \mathrm{E}(R_{z\mathbf{q}})$, and we can rewrite Equation (5):

$$\mathbf{E} = \mathrm{E}(R_{z\mathbf{p}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]\frac{\sigma_{\mathbf{p}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{p}} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]\frac{\sigma_{\mathbf{e}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{e}} \tag{6}$$

The intuition behind Equation (6) is straightforward. It is the CAPM where the efficient proxy portfolio is written as the sum of two portfolios: one that is inefficient and one that is the difference between an efficient portfolio and the inefficient one. For parsimony and without loss of generality, the efficient and inefficient portfolios have the same expected return.

Examining Equation (6) we identify three potential sources of misspecification that arise while using the CAPM with inefficient index portfolios. The first potential source of misspecification is, simply, ignoring the second addend of Equation (6).

The second potential source of misspecification is using incorrect excess expected return values due to the existence of portfolios that, although zero beta with respect to **p**, are of expected returns different than that of z**q**. If, then, in empirical tests, the latter portfolios are used, the excess expected returns values $[\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]$ are incorrect. We argue below that there are infinitely many such portfolios that could cause this misspecification. In particular, when this excess

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[15] Left multiplying Equation (1) by $\mathbf{a}^{\mathbf{T}}$ and rearranging yields $\sigma_{\mathbf{aq}} = \dfrac{\mathrm{E}(R_{\mathbf{a}}) - \mathrm{E}(R_{z\mathbf{q}})}{\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{z\mathbf{q}})}\sigma_{\mathbf{q}}^2$, which demonstrates the property if $\mathrm{E}(R_{\mathbf{a}}) = \mathrm{E}(R_{z\mathbf{q}})$.

expected return is zero, we say that the (inefficient) proxies induce zero relations. We examine these issues in the following sections.

The third potential source of misspecification is the use of unadjusted betas. Equation (6), the CAPMI, adjusts the CAPM betas by multiplying it by the ratio of the inefficient proxy's variance to the variance of a corresponding efficient proxy of the same expected return. This ratio is greater than one, and "becomes" one as the inefficient proxy "becomes" efficient. As this misspecification holds for all inefficient proxies, it agrees with the results of Ferguson and Shockley (2003) who find that, omitting debt, equity only (inefficient) proxies induce understated betas.[16]

We can rewrite Equation (6) such that it is additively separable in a traditional CAPM relation for **p** by writing the first addend without a beta adjustment. We accomplish that by recalling that $\sigma_{\mathbf{p}}^2 = \sigma_{\mathbf{q}}^2 + \sigma_{\mathbf{e}}^2$ (see above) and substituting for $\sigma_{\mathbf{p}}^2$ in Equation (6). We get

$$\mathbf{E} = \mathrm{E}(R_{\mathbf{zp}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{\mathbf{zp}})]\boldsymbol{\beta}_{\mathbf{p}} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{\mathbf{zp}})]\frac{\sigma_{\mathbf{e}}^2}{\sigma_{\mathbf{q}}^2}(\boldsymbol{\beta}_{\mathbf{p}} + \boldsymbol{\beta}_{\mathbf{e}}), \qquad (7)$$

where the first two addends on the right hand side are a traditional CAPM with respect to **p**.[17]

2.3 Where the CAPMI is Not Well Defined

In this section, we explore the case where the CAPMI is not well defined. This is the case where the proxy is of the same expected return as the GMVP. While willingly choosing a proxy of the GMVP expected return makes no sense, it is important to study this case because it is an empirical possibility, as the placements of the proxy, GMVP, and the other assets/portfolios in Markowitz world (the mean-

[16] Strictly speaking, the understatement is in the absolute value of the betas. Thus, for positive betas there is an understatement of the values and for negative ones there is an overstatement.
[17] We thank Richard Roll for suggesting this presentation.

variance space) are unobservable. Within this case, we further identify a special case, one where all securities have the same expected return.

Equation (6) implies that there is a zero coefficient of $\boldsymbol{\beta}_{\mathbf{p}}$ if and only if $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathrm{z}\mathbf{p}})$. The latter never happens with frontier zero beta portfolios because if $\mathrm{E}(R_{\mathbf{p}}) > \mathrm{E}(R_{\mathbf{GMVP}})$ $[\mathrm{E}(R_{\mathbf{p}}) < \mathrm{E}(R_{\mathbf{GMVP}})]$, then $\mathrm{E}(R_{\mathrm{z}\mathbf{p}}) < \mathrm{E}(R_{\mathbf{GMVP}})$ $[\mathrm{E}(R_{\mathrm{z}\mathbf{p}}) > \mathrm{E}(R_{\mathbf{GMVP}})]$ (where z**p** is a frontier portfolio). See, for example, Huang and Litzenberger (1988), Equation (3.14.2), which follows Merton (1972).[18] Also, geometrically, $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathrm{z}\mathbf{p}})$ (where z**p** is a frontier portfolio) requires a flat frontier tangent (parallel to the standard deviation axis), a situation that cannot happen.[19]

We will now examine the case where $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{GMVP}})$. Because the covariance of the GMVP with any security equals the variance of the GMVP,[20] it induces a beta of one all securities; there is no zero beta portfolio, zGMVP, and thus no zero beta rate; and we say that the CAPM is not well defined (with respect to the GMVP). We also note that as the reference frontier portfolio moves (along the frontier) toward the GMVP, the absolute value of the zero beta rate tends to infinity. When at least two securities have different expected returns, the CAPM relation does not exist. Geometrically, in this case, $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{GMVP}})$ implies a frontier tangent having no intersection with (and parallel to) the expected return axis.

If, however, all market securities have the same expected return, the frontier consists of one point only, which is also the GMVP, and any proxy has the same expected return as the GMVP. Thus, this is a special instance of the case described above where the CAPM is not well defined. Because all securities have the same

[18] Geometrically, this means that the above (below) GMVP frontier portfolios' tangent intersects the expected return axis below (above) the GMVP expected return.
[19] See the discussion of the case where all securities have the same expected return (below).
[20] See, for example, Roll (1997), also Huang and Litzenberger (1988), Section 3.12.

expected return and the same beta, and because the zero beta rate is not specified, there are infinitely many pairs of coefficients that average any constant (standing for the non-existent zero beta rate) and 1 (standing for any security's beta) to equal securities' expected return. In particular, there is a pair of coefficients that allows a theoretical zero relation: if the constant that stands for the (non-existent) zero beta rate is equal to securities' expected return, then a coefficient of one of the constant and a coefficient of zero of the betas explain all securities' expected returns. We call this a case of indeterminate degeneracy. We use the term *degeneracy* because expected returns degenerate to a single value, the hyperbola degenerates to a single point, the GMVP and the market portfolio degenerate to one portfolio, all betas degenerate to one, and a zero beta portfolio and thus the zero beta rate do not exist. We call this case *indeterminate* because there are infinitely many distinct pairs of coefficients that explain expected returns, of which the theoretical zero relation is only one. Because of the latter property, we also say that the theoretical zero relation is non-unique.

2.4 Discontinuity and Disparity

In a Markowitz world, there is an interesting "asymptotic discontinuity" when the reference portfolio becomes the GMVP. This discontinuity does not exist in a model with a risk-free asset. When there is a risk-free asset, the tangency portfolio becomes the GMVP as the risk-free rate goes to infinity (or negative infinity). Correspondingly, in analytical solutions, the weights of the frontier tangency portfolio go to the weights of the GMVP as the risk-free rate goes to infinity. Needles to say, the risk-free asset is always zero beta with respect to all risky portfolios, including the tangency portfolio.

In a zero beta model, as is the model in this paper, as the tangency portfolio tends to become the GMVP, the zero beta rate grows in absolute value and tends to infinity. However, as the tangency portfolio becomes the GMVP, its beta with any portfolio becomes one. There are no zero beta portfolios, and thus no zero beta rate.

Thus, in the "risk-free" case zero beta portfolios and a zero beta rate (albeit possibly infinitely high) always exist including the case where the tangency portfolio becomes the GMVP. In the zero beta case, in contrast, when the tangency portfolio becomes the GMVP, the beta it induces on all assets becomes one, there are no zero beta portfolios and no zero beta rate.

We call the phenomenon of "disappearance" of zero beta assets and rate within the zero beta model "asymptotic discontinuity" and the qualitative difference between the properties of the model with and without a risk-free rate "disparity."

2.5 Where the CAPMI is Well Defined

In this subsection we provide a very simple construction of zero relations and a hyperbola of zero beta portfolios for any inefficient proxy where the CAPM is well defined.[21] See Roll (1980) for a comprehensive study of zero beta portfolios existence and properties. We start the discussion with a numerical example that demonstrates the existence of, both, exogenous and endogenous zero relations. Exogenous zero relations arise between the original assets in a Markowitz world. Note that in a Markowitz world, there is no restriction on the number of original assets that are uncorrelated. This number could be zero, two, or equal to the number of all original assets (diagonal covariance matrix.). Endogenous zero relations arise between assets or portfolios and portfolios, which are not originally uncorrelated. Thus, an interpretation of this example should be that in a Markowitz world there is no limit to the number of cases similar to those in the example because of potential existence of exogenous zero relations.

[21] For simplicity, we do not repeat the phrase, "where the CAPM is well defined" through the section.

Numerical example. Assume a four assets, **q**, **p**, **u**, and **v**, Markowitz world. If for $\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \\ \mathbf{u} \\ \mathbf{v} \end{pmatrix}$, we have $\mathbf{E} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then, solving for the portfolio frontier identifies **q** and **v** as frontier portfolios. Thus, we can view **p** as some inefficient proxy and note that **u** induces a zero relation with respect to **p** as it is uncorrelated with it and has the same expected return. More specific structure to support this example could be as follows. Because **q**, **p**, and **u** have the same expected return, projecting **p** and **u** on **q** yields $\mathbf{p} = \mathbf{q} + \boldsymbol{\varepsilon}_{\mathbf{p}}$, and $\mathbf{u} = \mathbf{q} + \boldsymbol{\varepsilon}_{\mathbf{u}}$, respectively, where both $\boldsymbol{\varepsilon}_{\mathbf{p}}$ and $\boldsymbol{\varepsilon}_{\mathbf{u}}$ are of mean zero and uncorrelated with **q**. Then, setting $\sigma_{\boldsymbol{\varepsilon}_{\mathbf{p}}\boldsymbol{\varepsilon}_{\mathbf{u}}} = -\sigma_{\mathbf{q}}^{2}$ implies $\sigma_{\mathbf{pu}} = 0$. Thus, **u** is a zero beta portfolio of and with the same expected return as **p**, inducing a zero relation. This could be the case, for example, where the **q**, **p**, **u**, and **v** are distributed according to a multivariate normal distribution. As **p** and **u** are original exogenously given assets we call the zero relation that **u** induces with respect to **p** an exogenous one.

The intuition behind the existence of exogenous zero relation portfolios as **p** and **u** in the example above and in general is straightforward. It follows from the property that a Markowitz world specifies the first two moments of return distributions, leaving freedom to further specify "distributions structure." In order to leave "other things equal," a constraint on such "distribution structuring" is that it should not change the frontier.

We will now demonstrate that, within the example's Markowitz world, there is an endogenously determined asset, a combination of **p** and **q**, where **p** and **q** are positively correlated, which induces a zero relation with respect to **p**. This asset, say z**p**, has a weight of 2 in **q** and -1 in **p**. Thus, the variance of z**p** and its covariance with **p** are

$$\sigma_{\mathbf{zp}}^{2}=\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}1&1&1&0\\1&2&0&0\\1&0&3&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}=2\text{, and }\sigma_{\mathbf{pzp}}=\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}1&1&1&0\\1&2&0&0\\1&0&3&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=0.$$

As the expected returns of **p** and z**p** are equal and they are uncorrelated, z**p** induces a zero relation with respect to **p**.

We will now show that the latter property is not coincidental to the last example but, in fact, is a general property in this context: it exists for any inefficient proxy at any Markowitz world. Consider some inefficient proxy, **p**, and the frontier portfolio with the same expected return **q**. Consider now the (degenerate) hyperbola spanned by **q** and **p** only. We claim that on this single expected return hyperbola, **q** must be the GMVP. This is because **q** was already the GMVP for its expected return on a hyperbola that was spanned by **q**, **p** and additional assets. The removal of the additional assets from the set of assets available to span the hyperbola, could not have improved the optimum, that is, could not have allowed the creation of a portfolio with variance lower than that of **q**.[22] Thus, **q** must still be the GMVP on the hyperbola spanned by **q** and **p**.

It is a well known property that a GMVP's covariance with all assets is equal to a positive constant, its variance [see Huang and Litzenberger (1988), Section. 3.12, for example].[23] This property, together with the one that we demonstrated above, that

[22] Presence of additional assets is not necessary for the argument, of course. However, were there no additional assets, **q** would have been the GMVP of the original hyperbola.
[23] This property must also follow, and indeed follows, from direct calculation of the covariance between the GMVP and any portfolio **a**. The (weight vector of the) GMVP [see, for example, Feldman and Reisman (2003)] is $\frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}$. Thus, the covariance of the GMVP with any portfolio **a** is

q is the GMVP on the hyperbola spanned by **q** and **p**, imply that within any Markowitz world any inefficient proxy **p** and the frontier portfolio of the same expected return, **q**, have a covariance matrix of the form $\begin{pmatrix} v_F & v_F \\ v_F & v_I \end{pmatrix}$, $v_I > v_F > 0$, where v_F is the variance of the frontier portfolio **q**, and v_I is the variance of the inefficient portfolio of the same expected return, **p**.[24] It, thus, becomes straight forward to identify a pair of weights, $(\alpha, 1-\alpha)$, of a portfolio that combines **q** and **p**, respectively, and form a portfolio that is uncorrelated with **p**. The weights of such a portfolio must solve the equation $\begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} v_F & v_F \\ v_F & v_I \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. Solving the equation we get the well defined solution, $(\alpha, 1-\alpha) = \left(\frac{v_I}{v_I - v_F}, -\frac{v_F}{v_I - v_F} \right)$. Note that the weight of the frontier portfolio, $\frac{v_I}{v_I - v_F}$, is always positive and greater than one. It is the ratio of the variance of the inefficient portfolio over the variance increment of the inefficient portfolio over the frontier portfolio's variance. This ratio can be interpreted as related to a relative measure of inefficiency. We also note that the variance of the zero relation portfolio is $\frac{v_I v_F}{v_I - v_F}$ $\left[\text{because, } \sigma^2_{\mathbf{zp}} = \begin{pmatrix} \frac{v_I}{v_I - v_F} \\ \frac{-v_F}{v_I - v_F} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} v_F & v_F \\ v_F & v_I \end{pmatrix} \begin{pmatrix} \frac{v_I}{v_I - v_F} \\ \frac{-v_F}{v_I - v_F} \end{pmatrix} = \frac{v_I v_F}{v_I - v_F} \right]$.

We identify additional properties. Where $v_I = 2v_F$, as is the case in our numerical

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$\mathbf{a}^{\mathrm{T}}\mathbf{V}\frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}} = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}$, a positive constant, independent of **a**. As **a** could stand for the GMVP, this covariance is also the variance of the GMVP.

[24] This property is also implied by the CAPM relation. Rearrange the CAPM relation for some portfolio **p** with respect to some non-GMVP frontier portfolio **q**, as $\sigma^2_{\mathbf{q}} \frac{\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{q}})}{\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{z\mathbf{q}})} = \sigma_{\mathbf{pq}}$. Thus, for any portfolio **p** with the same expected return as **q**, this relation becomes $\sigma^2_{\mathbf{q}} = \sigma_{\mathbf{pq}}$. In particular, for any **p** and **u** that have the same expected return as **q** and possibly $\sigma^2_{\mathbf{p}} \neq \sigma^2_{\mathbf{u}}$, applying the above relation twice, we have $\sigma^2_{\mathbf{q}} = \sigma_{\mathbf{pq}} = \sigma_{\mathbf{uq}}$.

example above, the variance of the zero relation portfolio will be equal to the variance of the inefficient proxy, that is, $\sigma^2_{\mathbf{p}} = \sigma^2_{\mathbf{zp}}$. (Of course, the expected returns of these portfolios are equal as well.) Further, $v_I > 2v_F$ $(v_I < 2v_F)$, implies $\sigma^2_{\mathbf{zp}} < 2v_F$ $(\sigma^2_{\mathbf{zp}} > 2v_F)$.

If we define a measure of relative inefficiency *RI*, $RI \triangleq \dfrac{v_F}{v_I - v_F}$, we can write the variance ratio of the zero relation portfolio return over the frontier portfolio return as $\dfrac{\sigma^2_{\mathbf{zp}}}{v_F} = \dfrac{v_I}{v_I - v_F} = 1 + RI$. Then, we note that $\dfrac{\sigma^2_{\mathbf{zp}}}{v_F} = 1 + RI \xrightarrow[RI \to 0]{} 1$, that is, as the "inefficiency" of the portfolio proxy grows, the zero relation portfolio gets closer to the frontier, and conversely, $\dfrac{\sigma^2_{\mathbf{zp}}}{v_F} = 1 + RI \xrightarrow[RI \to \infty]{} \infty$, that is, the closer the portfolio proxy gets to the frontier, the higher is the variance of zero relation portfolio.

Graphical representations of the Numerical example. We will now present eight graphs that manifest the numerical example. Figure 1 depicts the market's four assets **q**, **p**, **u**, and **v**, the portfolio frontier they induce, and the GMVP.

Figure 1: The Portfolio Frontier

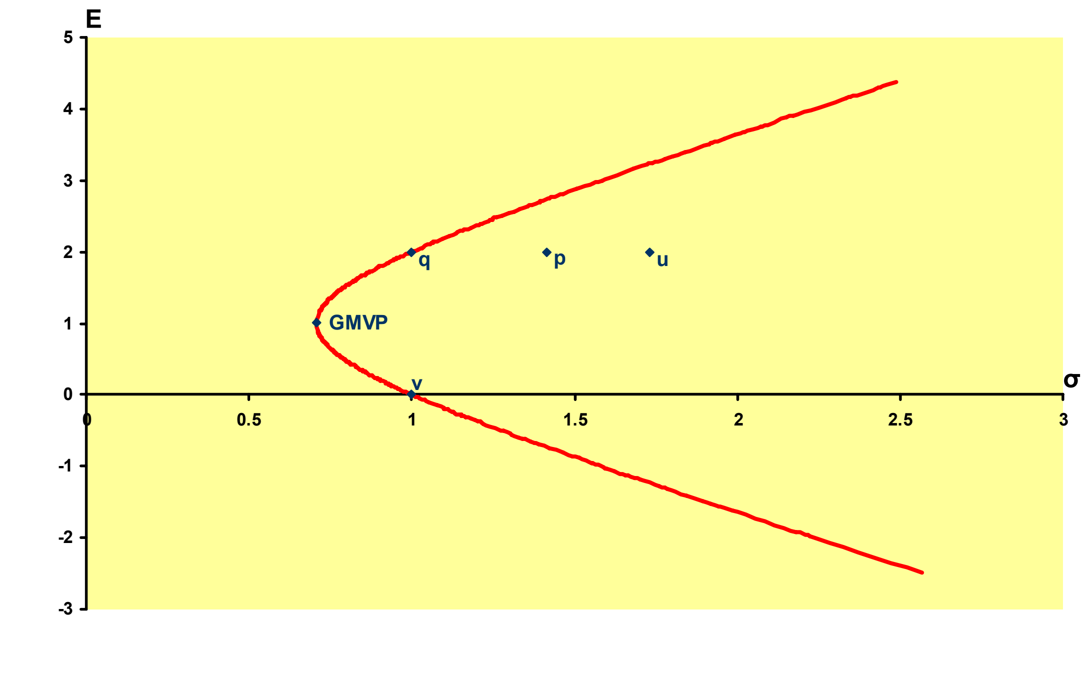


Figure 2 depicts the tangent to the efficient proxy **q,** which is also the pricing line induced by **q**. Note that **v** is a zero beta portfolio to **q** as it is at the level of the intersect of the tangent, or on the horizontal line.

Figure 2: The Tangent Line and The Zero-Beta Portfolios

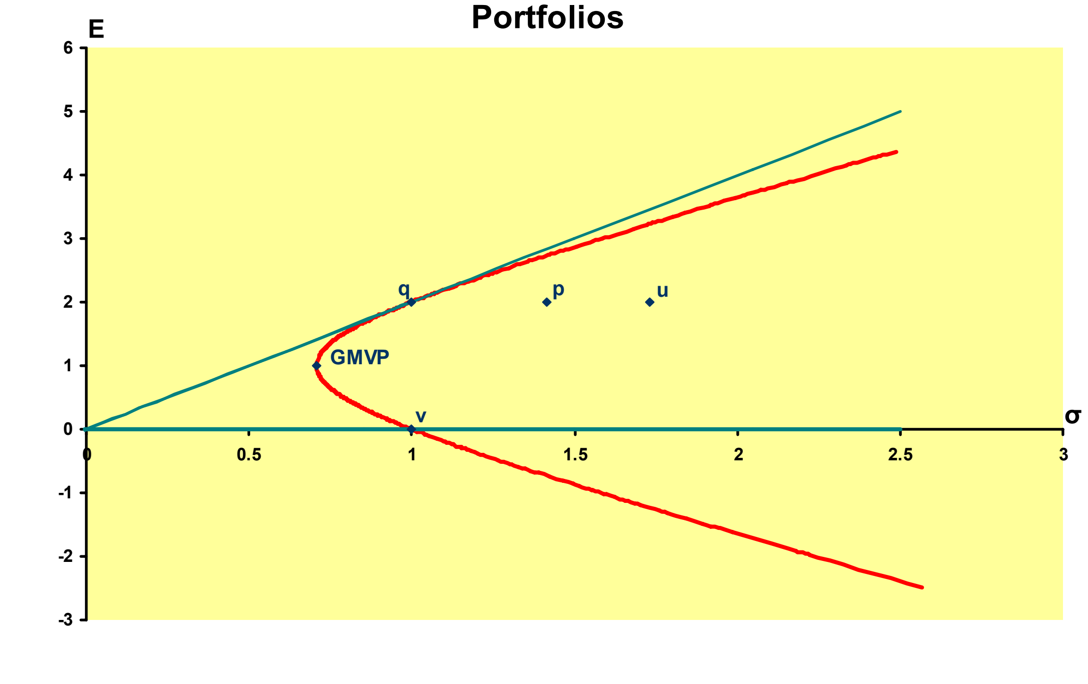


Figure 3 depicts the hyperbola spanned by **p** and GMVP. This hyperbola must have GMVP as its own GMVP.

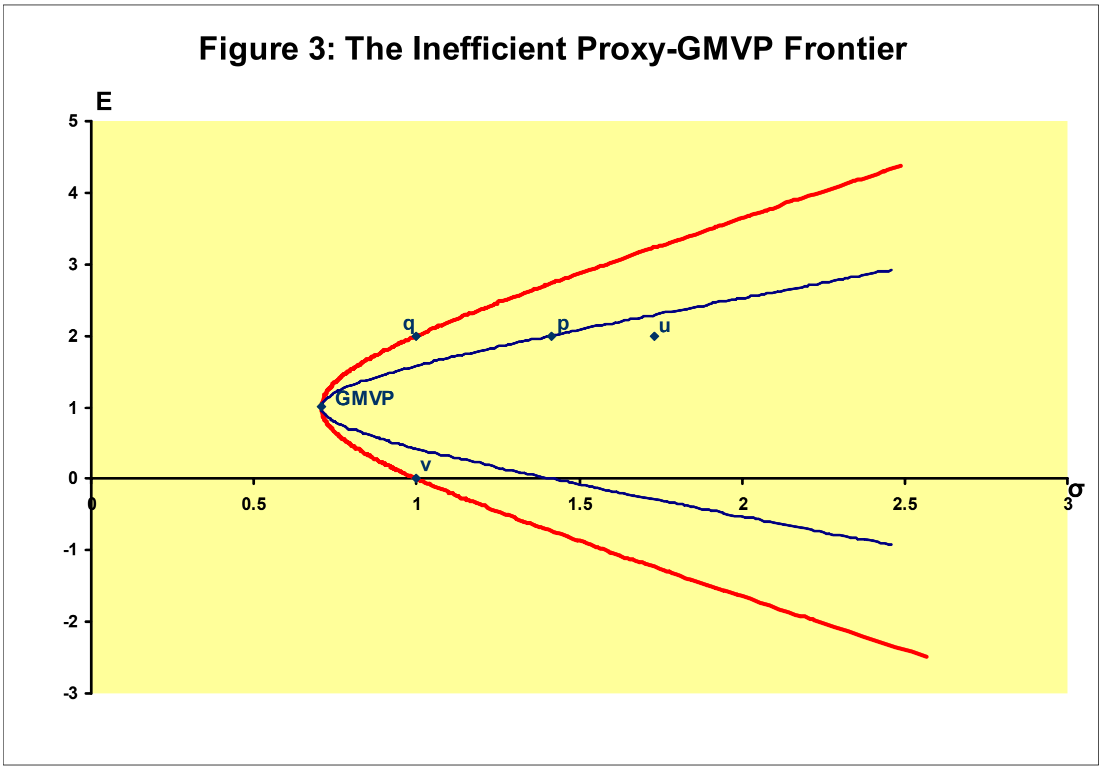


Figure 4 depicts the tangent to **p** on the hyperbola spanned by **p** and GMVP. This tangent defines z**p**, **p**'s zero beta portfolio on this hyperbola and a locus of higher variance zero beta portfolios to **p** at the expected return of z**p,** on the green line.

Figure 4: The Inefficient Proxy's Minimum Variance Zero-Beta Portfolios

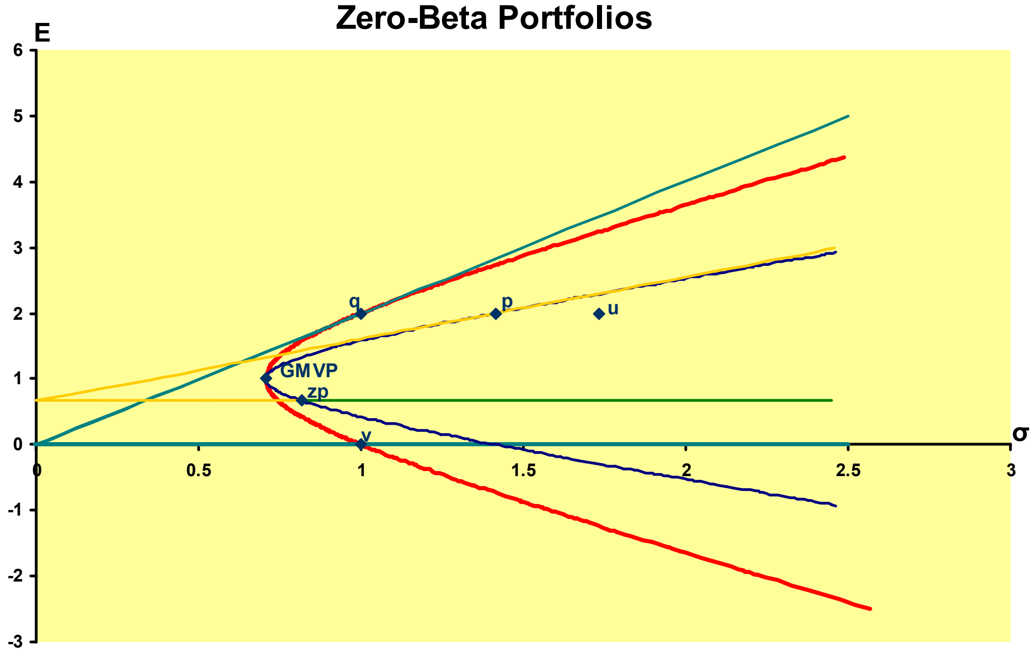


Figure 5: The Inefficient Proxy's Endogenous Zero-Beta Frontier

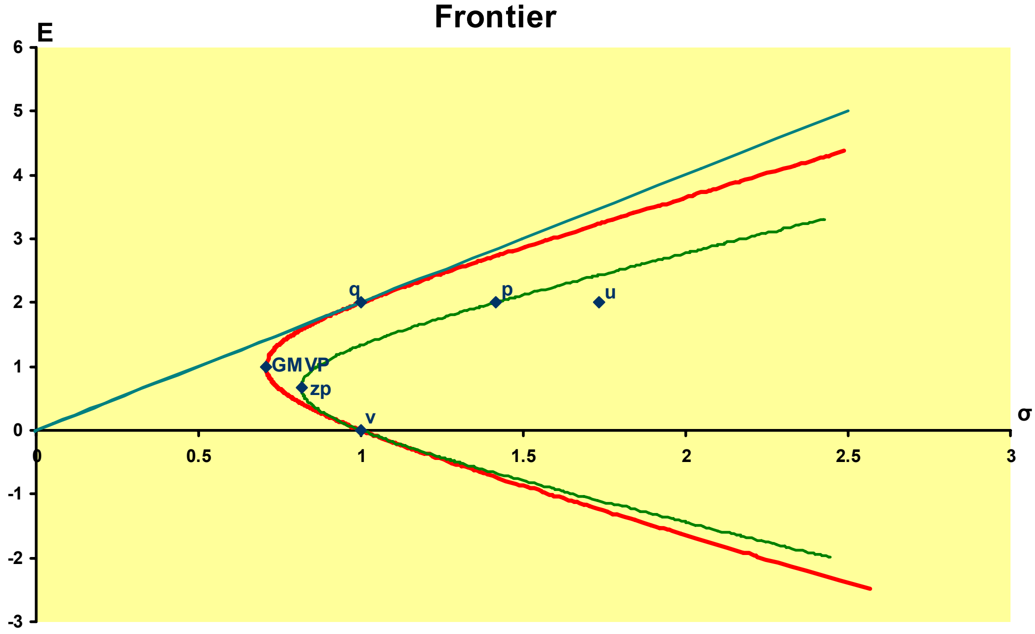


Figure 5 depicts the hyperbola spanned by **v** and z**p**. As both spanning portfolios are zero beta with respect to **p,** all this hyperbola's portfolios are also zero beta with respect to **p**. In our example, this hyperbola goes through the expected value and standard deviation coordinates of **p**. As we demonstrate above, this is a special case that occurs when the variance of the inefficient proxy, **p**, is double that of the corresponding efficient one, **q**. The analysis above also demonstrates that if **p** "moves" to the left (right), the hyperbola moves to the right (left). Note that this frontier/hyperbola is the zero is the locus of the minimum variance zero beta portfolios of **p**. Thus, for example, all exogenous zero relation portfolios, induced by **u**, for example, will be contained within this hyperbola [see Roll (1980)].

Figure 6 superimposes Figure 5 on Figure 4 and depicts two loci of portfolios that are zero beta with respect to **p**: the horizontal line that passes through z**p** and the hyperbola spanned by **v** and z**p**. Combinations of portfolios from each locus further induce loci of portfolios that are zero relation portfolios with respect to **p**.

Figure 6: The Inefficient Proxy's Zero-Beta Portfolios

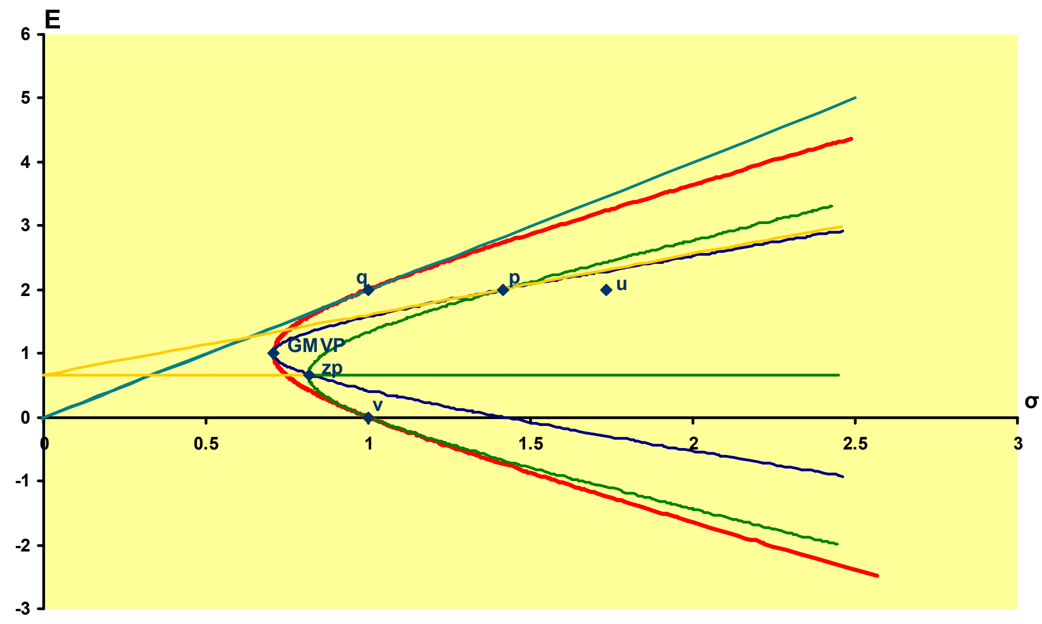


Figure 7 depicts the direct generation of a zero relation to **p** by combining **p** and **q**. As in the analysis above and in Figure 5, the zero relation portfolio to **p**, in our example, has the same expected value and standard deviation as **p**.

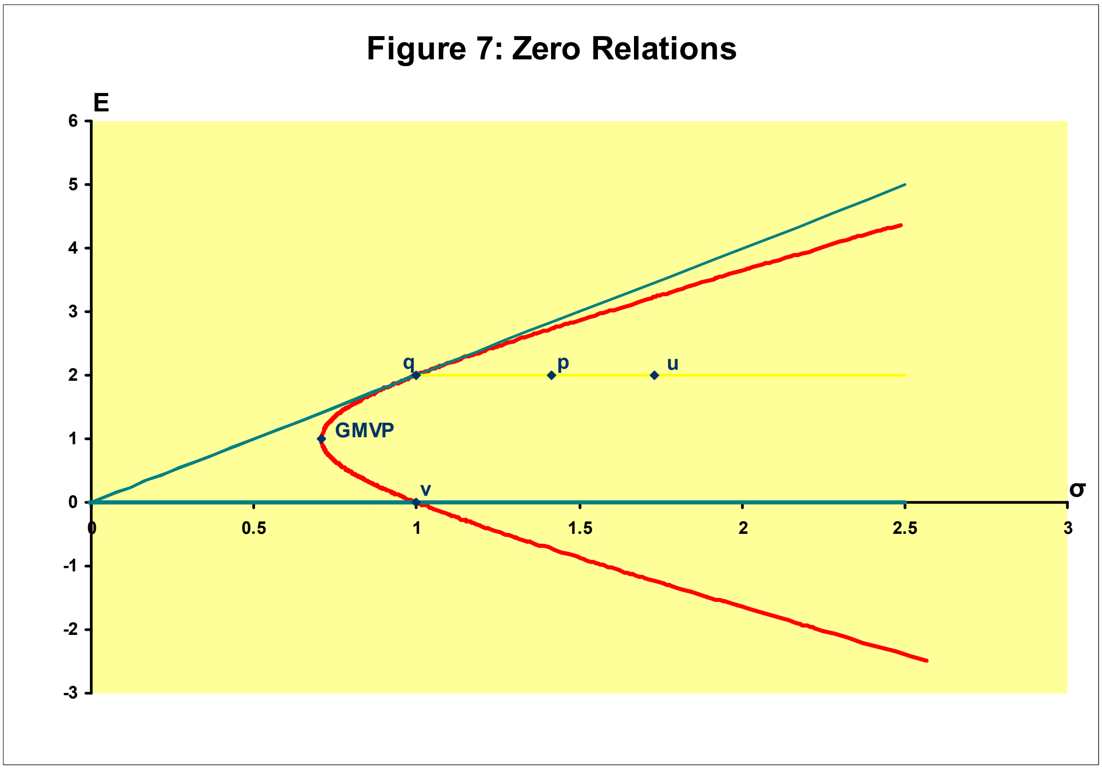
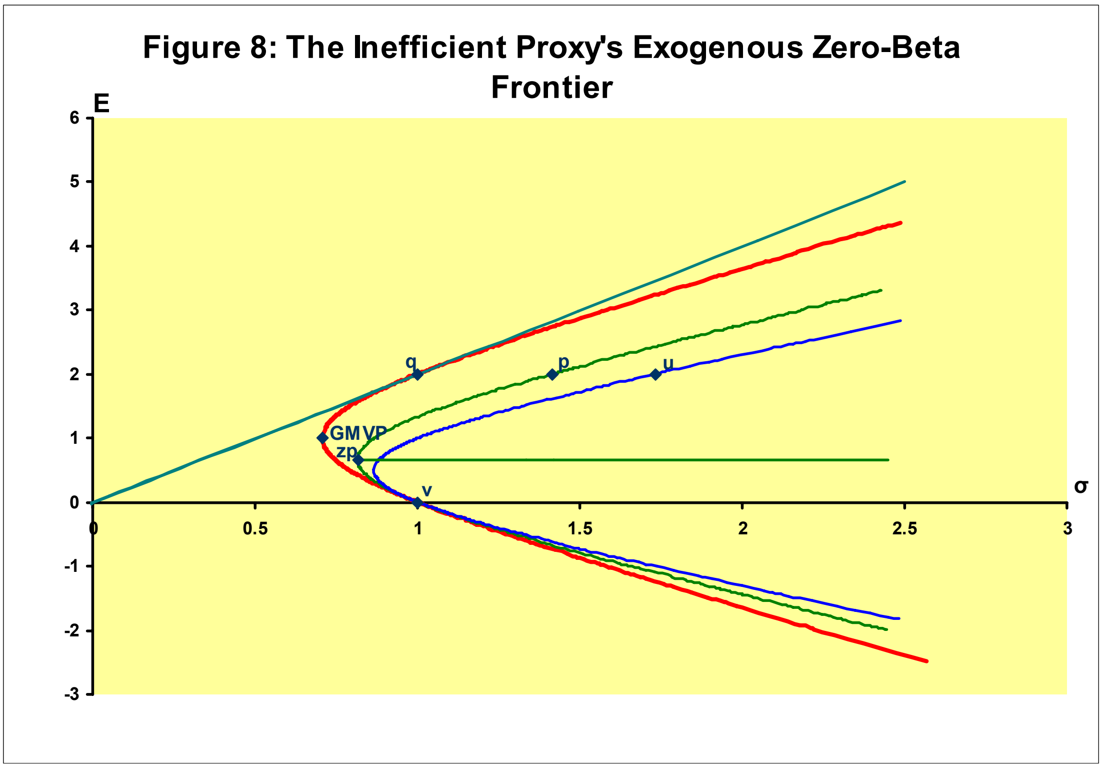


Figure 8 depicts an additional locus of portfolios that are zero beta with respect to **p**, generated by portfolio **u**, a market portfolio that is uncorrelated with **p**.



We have thus, proved and illustrated the following proposition and corollary.

Proposition 1. *i) In a Markowitz world, any inefficient proxy induces a zero relation. ii) Let, without loss of generality, the variance of some inefficient portfolio proxy,* **p**, *be* v_I *and that of the frontier portfolio of the same expected return,* **q**, *be* v_F, $v_I > v_F > 0$. *Then, the portfolio whose weights are* $\left(\frac{v_I}{v_I - v_F}, \frac{-v_F}{v_I - v_F}\right)$ *in* $(\mathbf{q}, \mathbf{p})$, *respectively, induces a zero relation with respect to* ***p****, and its variance is* $\frac{v_I v_F}{v_I - v_F}$.

Corollary 1. *If the variance of the inefficient proxy is double that of the frontier portfolio of the same expected return, then, the zero relation portfolio has the same variance (and, of course, the same expected return) as that of the inefficient proxy. As the inefficient portfolio proxy gets closer to the frontier, the variance of its zero relation grows to infinity. Conversely, as the variance of the inefficient portfolio proxy grows to infinity, its zero relation portfolio gets closer to the frontier.*

The following proposition identifies, for any inefficient proxy, a zero beta portfolio at a different expected return than that of the inefficient proxy and its zero relation portfolio that was identified in Proposition 1. It is the minimum variance inefficient proxy's zero beta portfolio among all of the inefficient proxy's zero beta portfolios at all expected returns.

Proposition 2*. [Roll (1980), Huang and Litzenberger (1988), Section 3.15]. Consider the hyperbola spanned by some inefficient proxy and the GMVP. Then, the GMVP is the GMVP of this hyperbola as well, and the zero beta portfolio of the inefficient proxy on this hyperbola, is the minimum variance zero beta portfolio of the inefficient proxy, among all the zero beta portfolios of the inefficient proxy.*

The proof of the first part of Proposition 2 is similar to the proof of Proposition 1. The proof of the second part of Proposition 2, the identification of the inefficient proxy's zero beta portfolio as the minimum variance one among all its zero beta portfolios, is demonstrated in Huang and Litzenberger (1988), Section 3.15, by Lagrange's method.

Corollary 2. *The zero beta portfolios, with respect to some inefficient proxy, identified in Propositions 1 and 2, are of different expected returns.*

Proof. The zero beta / zero relation portfolio identified in Proposition 1 is of the same expected return as the inefficient proxy. The zero beta portfolio identified in Proposition 2 is on the other side, with respect to the inefficient proxy, of the (non degenerate) hyperbola spanned by the inefficient proxy and the GMVP [see, for example, Huang and Litzenberger (1988), Section 3.15]. Thus, they must be of different expected returns.

As the two zero beta portfolios identified in the propositions above are of different expected returns, they span a zero beta hyperbola that extends to all expected returns. We state this property in the following proposition.

Proposition 3. *Any inefficient proxy induces a hyperbola of zero beta portfolios that extends to all expected returns. Such a hyperbola is the one spanned, for example, by the zero relation portfolio identified in Proposition 1, and by the "minimum variance zero beta portfolio" identified in Proposition 2. Moreover, this hyperbola consists of the minimum variance zero beta portfolios at every expected return. The hyperbola includes one frontier portfolio, the (single) frontier portfolio that is uncorrelated with the frontier portfolio that has the same expected return as the inefficient proxy.*

Roll (1980) attains the results of Proposition 3 in a different way. Using our approach, the proof of the first and second part of Proposition 3 is straight forward. Proving the latter part of the proposition, the property that the said hyperbola consists of the minimum variance zero beta portfolios for each expected return can be done by contradiction. Following the proof of Proposition 1, existence of a zero beta portfolio with lower variance than that of the said hyperbola portfolio, will facilitate combining it with the frontier portfolio of the same expected return and constructing a portfolio with variance lower than that of the frontier portfolio. This is, of course, a contradiction.

Note, also, that any zero beta hyperbola includes a single frontier portfolio. This frontier portfolio is the (only) frontier portfolio that is uncorrelated with the frontier portfolio of the same expected return as that of the inefficient proxy that induces the zero beta hyperbola. In fact, all portfolios of the same expected return are uncorrelated with a single frontier portfolio. On the other hand, all the portfolios uncorrelated with a frontier portfolio are of a single expected return. A consequence of this is that as an inefficient proxy becomes efficient, the zero beta hyperbola it induces degenerates/collapses to a degenerate (single expected return) hyperbola (or a line). See Roll (1980).

We reemphasize that although a zero relation generally induces a zero R^2 in a CAPM type regression, the choice of any zero beta portfolio at any expected return—except the single expected return corresponding to the frontier zero beta portfolio with respect to the proxy (z**q** in our case)—induces a pricing error by inducing an incorrect excess expected return value / risk premium / coefficient on the beta in the CAPM relation. As we demonstrated, there are infinitely many such portfolio and for every expected return. The likelihood of identifying the “correct” zero beta portfolio among the infinitely many ones seems to be negligible.

2.6 The CAPMI Market Model: Correlated Explanatory Variables

We cannot say that the omitted addend is uncorrelated with or orthogonal to, the existing addends.

Following Sharpe (1963) and Black (1972), we can write the CAPMI market model. To do that, we replace, the explanatory random variable $R_{\mathbf{q}}$ in the CAPM market model with the difference in the random variables $R_{\mathbf{p}} - R_{\mathbf{e}}$.[25] Thus, we replace one market model addend, related to $R_{\mathbf{q}}$, with two addends, related to $R_{\mathbf{p}}$ and $R_{\mathbf{e}}$ respectively. Recalling the construction method of the CAPMI, it is easy to see that if **p** is indeed an inefficient portfolio (that is, if $R_{\mathbf{p}} \neq R_{\mathbf{q}}$), then $R_{\mathbf{p}}$ and $R_{\mathbf{e}}$ must be correlated. In other words, the two "new" addends in the CAPMI market model must be correlated. This property might be material when considering the misspecification caused by ignoring, in implementations and tests, the addend related to $R_{\mathbf{e}}$. We cannot say that the omitted addend is orthogonal to, or uncorrelated with, the existing addends.

3 Implications

In this section we list a few implications of a Markowitz world.

3.1 Misspecification of the CAPM and a Reemphasis of the Roll and Ross, Kandel and Stambaugh, and Jagannathan and Wang Implication

Equation (6) is a well-specified CAPMI and is distinctly different from the CAPM.[26] We say that when using inefficient proxies with the CAPM, we use a misspecified relation because we unjustifiably and incorrectly force an addend in the specified equation to be zero. This misspecification reemphasizes the important RR, KS, and Jagannathan and Wang results that demonstrate that it is meaningless to use inefficient proxies to implement regressions of CAPM, which is designed to use efficient proxies. For example, KS write in their abstract, "If the index portfolio is inefficient, then the coefficients and R^2 from an ordinary least squares regression of

[25] Recall that by construction $R_{\mathbf{p}} = R_{\mathbf{q}} + R_{\mathbf{e}}$, thus $R_{\mathbf{q}} = R_{\mathbf{p}} - R_{\mathbf{e}}$.

[26] As specified in Equation (1), for example.

expected returns on betas can equal essentially any values…." Because real-world proxies are practically inefficient, such regressions based on the classical CAPM are misspecified. Jagannathan and Wang (1996, p. 41), provide an example of a portfolios rearranging, to which the CAPM should not be sensitive, that reduces the R^2 from 95% to zero.

The misspecification that we demonstrate is robust with respect to the explanatory power of the betas. Positive explanatory power of the betas does not imply that the well-specified CAPMI would have resulted with the same values for R^2 and coefficients. In other words, CAPM regressions that unduly constrain a specification addend to be zero are subject to getting meaningless R^2 and coefficients' values regardless of the R^2 and coefficients they produce. Thus, CAPM regressions that use different procedures from those used by Fama and French (1992), and that are able to produce positive beta explanatory power, are also subject to the same misspecification. In addition, this misspecification is robust to multiperiod and multifactor models, and to those conditioning on various attributes.

A multitude of CAPM empirical studies followed the introduction of the CAPM in Sharpe (1964), Lintner (1965), Mossin (1966), and Black (1972), and the seminal empirical works of Black, Jensen, and Scholes (1972) and Fama and Macbeth (1973). Curiously, however, the issue of the misspecification with respect to inefficient proxies though highlighted by Roll's critique, Roll (1977), was largely ignored, and was not attended to until the Fama and French (1992) results induced the declaration, "Beta is dead…". A notable exception is Lehmann (1989), who developed cross-section efficiency tests acknowledging important property of inefficient proxies: the inducement of zero beta portfolios at all expected returns.

3.2 Infinitely Many Theoretical Zero Relations Within a Markowitz World

While the main implication of this paper is the misspecification of the CAPM for inefficient portfolios and the values of the misspecified coefficients and R^2 are immaterial, the prevalence and likelihood of zero relations has captured special interest in the literature. RR said, in their abstract, "For the special case of zero relation, a market portfolio proxy must lie inside the frontier, but it may be close to the frontier." On page 104, they write, "Portfolios that produce a zero cross-sectional slope…lie on a parabola that is tangent to the efficient frontier at the global minimum variance point." In addition, their Figure 1, page 105[27] draws a boundary region that contains zero relation proxies, one such portfolio being 22 basis points away from the portfolio frontier. We emphasize that where the CAPM is well defined, any inefficient proxy has at least one and possibly infinitely many portfolios that induce zero relations.

We say that for proxy portfolios whose expected returns are equal to that of the GMVP, the CAPMI is not well defined because, as described above, the GMVP has no zero beta portfolio and the limit zero beta rate is infinity. We identify, however, a degenerate indeterminate case that non-uniquely allows a theoretical zero relation: where all securities have the same expected return. The theoretical zero relation, however, is one possible relation out of infinitely many possible ones.

3.3 The Misspecification with Respect to Any Zero Beta Portfolio

When considering the misspecification of the CAPM for inefficient proxies where the CAPM is well defined, it is important to note that zero beta portfolios other than those noted below induce an incorrect excess expected return premium in the CAPM relation and, thus, a pricing error. This is in addition to the zero beta portfolios

[27] This figure is reproduced as Figure 13.1, in Bodie, Kane, and Marcus (2005), Chapter 13, page 421.

with expected returns equal to that of the proxy, which induce zero relations, and in addition to the zero beta portfolios of expected return equal to that of the frontier zero beta portfolio, which induce the correct excess expected return value in the CAPM relation. As stated above, there are infinitely many such portfolios and for each expected return.

We can specify regions where the zero beta portfolios could lie [see Roll (1980)], but considering the measure of these portfolios out of all portfolios might be irrelevant. Also, because a Markowitz world specifies only the first two moments of assets' return distributions, each point in the mean-variance space might represent more than one asset. These zero beta portfolios induce zero relations or incorrect excess expected return values, thus, pricing errors.

3.4 A Robust CAPMI and Incomplete Information Equilibria

Expected returns and variances, and thus the portfolio frontier, are unobservable. Moreover, assets that are correlated with returns on optimally invested wealth or consumption growth—human capital, real-estate, and energy for example—are not fully securitized and traded. Thus, in all likelihood, real-world portfolio proxies are inefficient. Though Equation (6) is a robust CAPMI in the sense that it holds for all proxy portfolios whether efficient or inefficient, an interesting question might arise regarding the usefulness of this relation, as inefficient proxies are unobservable as well. The answer to this question is twofold. First, observable or unobservable, the CAPMI had better be well specified. Particularly, the CAPMI expresses any portfolio as a combination of an inefficient one and the difference between an efficient and the inefficient one. The CAPM constrains this difference to be zero, limiting portfolios to be efficient. This constraint, however, is not satisfied; thus, the CAPM, which is a constrained (special case) of the CAPMI, is misspecified.

Because the CAPM is misspecified for inefficient portfolios, we should use the CAPMI in implementation and testing.

Second, to resolve the problem of unobservable means and covariances, we suggest the use of an incomplete information methodology. There we would identify a CAPM in terms of endogenously determined moments. We would use Bayesian inference methods (filters) to form these moments, conditional on observations. These observations would include (noisy) functions of the sought after moments, such as prices, outputs, and macroeconomic variables. Such equilibria in a multiperiod, multifactor context were developed by Dothan and Feldman (1986), Detemple (1986, 1991), Feldman (1989, 1992, 2002, 2003), Lundtofte (2006, 2007), and many others. Feldman (2005) includes a review of incomplete information works and a discussion of issues related to these equilibria.

4 Conclusion

The Sharpe-Lintner-Mossin-Black classical CAPM type relation (CAPM) implies an exact non-zero relation between expected returns and betas of frontier portfolios other than the GMVP. Because neither expected returns nor betas are directly observable and because not all assets that covary with the return on optimally invested portfolios or consumption growth are fully securitized, it is highly likely that CAPM implementations and tests use inefficient portfolios proxies. Roll and Ross (1994), Kandel and Stambaugh (1995), and Jagannathan and Wang (1996) in seminal works, demonstrate that inefficiency of proxy portfolio might render CAPM regression results meaningless. They offer their finding as the reason behind the empirical results of Fama and French (1992) and others, and they intensively examine the relation between expected returns and betas.

We complement the RR and KS findings by specifying the CAPMI, the CAPM relation for any (inefficient) portfolio. We suggest that because we use inefficient proxies, we should use the CAPMI in implementations and tests, and not use the CAPM, which is misspecified for use with inefficient portfolios. Three sources of misspecification arise when using the CAPM with inefficient index portfolios. One source of misspecification stems from ignoring an addend in the CAPMI. The second source arises because of the infinitely many zero beta portfolios, at all expected returns, which are likely to induce incorrect excess expected return values in the CAPM relation. And the third source of misspecification arises because betas of inefficient proxies are different from those of efficient ones.

Using the CAPM with inefficient proxies is a misspecification that renders the resulting coefficients and R^2 meaningless. This reemphasizes the RR and KS implication that the CAPM is misspecified for use with inefficient proxies, which renders CAPM regressions with inefficient proxies meaningless. This misspecification is robust to CAPM procedures that, unlike Fama and French (1992), find explanatory powers of betas and is robust to various extensions of the basic model, such as multiperiod, multifactor, and the conditioning on various attributes. To overcome the problem that means and covariances are not observable, we suggest implementing and testing incomplete information equilibria, described in Feldman (2005), for example.

While the analysis in this paper is done in a single period mean-variance framework, its implications apply to multiperiod, multifactor models. This is because we can see the single period mean-variance model here as a “freeze frame” picture of a dynamic equilibrium where, because of the tradeoff between time and space, only the instantaneous mean and instantaneous variance of returns are relevant until the next decisions' revision in the next time instant.

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