Stochastic Modeling of Private Equity – An Equilibrium Based Approach to Fund Valuation

Axel Buchner[*], Christoph Kaserer[†] and Niklas Wagner[‡]

Center for Entrepreneurial and Financial Studies (CEFS)
and Department for Financial Management and Capital Markets

Technische Universität München
Arcisstr. 21
D-80290 München

www.cefs.de
www.ifm.wi.tum.de

axel.buchner@wi.tum.de
christoph.kaserer@wi.tum.de
niklas.wagner@wi.tum.de

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[*]Dipl.-Kfm. Axel Buchner, Tel: +49 - 89 - 289 25485, Fax: +49 - 89 - 289 25488
[†]Prof. Dr. Christoph Kaserer, Tel: +49 - 89 - 289 25489, Fax: +49 - 89 - 289 25488
[‡]PD Dr. Niklas Wagner, Tel: +49 - 89 - 289 25486, Fax: +49 - 89 - 289 25488

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Abstract

In this paper, we present a new approach to measure the returns of private equity investments based on a stochastic model of the dynamics of a private equity fund. Our stochastic model of a private equity fund consists of two independent stages: the stochastic model of the capital drawdowns and the stochastic model of the capital distributions over a fund's lifetime. Capital distributions are assumed to follow lognormal distributions in our approach. A mean-reverting square-root process is applied to model the rate at which capital is drawn over time. Applying equilibrium intertemporal asset pricing considerations, we are able to derive closed-form solutions for the market value and time-weighted model returns of a private equity fund.

Keywords:
Private Equity Funds, Stochastic Modeling, Mean-Reverting Square-Root Process, Incomplete Markets.

JEL classification code: G24, D52, G13

1 Introduction

Illiquid assets, such as private equity, constitute a significant portion in the long-term strategic asset allocations of many institutional investors. According to figures provided by the EVCA, annual funds raised by private equity and venture capital management companies in Europe reached a record high of Euro 71.8 bn in the year 2005. This figure represents more more than two and a half times the funds raised in 2004 of Euro 27.5 bn, which emphasizes the importance private equity has gained especially in the recent years.

The illiquid character of investments in private equity as an asset class presents particular challenges for portfolio management. The empirical application of the standard neoclassical financial models - such as portfolio optimization techniques and value at risk - typically require as model inputs the risk-return characteristics of an asset class. Risk-return characteristics for asset classes of publicly traded securities can easily be estimated by standard statistical procedures from historical time-weighted returns based on the securities observable market prices. As private equity investments are not traded on secondary markets, observable market prices are constantly not available. The evaluate the performance of private equity funds the typical measure used is therefore the internal rate of return (IRR) of the investment. As has been intensively discussed in the literature, the use of the IRR as a measure of performance has several drawbacks. According to Hirschleifer (1970), the IRR may not be unique when future cash flows vary in sign. Second, the IRR is based on the implicit assumption that intermediate cash flows can be reinvested at the discount rate. Last, but maybe most important in the context of portfolio management is the fact that the IRR does not allow the estimation of a standard deviation of returns and a correlation of private equity returns to other asset classes, such as publicly traded stocks. Some empirical studies on the performance of private equity funds try to avoid these drawbacks of the IRR by calculating time-weighted returns based on the fund's disclosed net asset values (NAV).[1] These time-weighted returns are based on the implicit assumption that the assets of the fund may be realized (or are at least accurately measured) by the reported net asset values of the fund management. However, as discussed in the relevant literature, reported net asset values frequently suffer from the problem of *stale* and *managed* pricing.[2] Stale pricing is caused by the fact that the reported net asset values of the fund management do not readily incorporate all available information. Stale pricing hence leads to a lag-time between observable market valuations and valuations in private equity portfolios. Under these conditions, the reported net asset values will only occasionally reflect the true market values, i.e. the price at which the fund's asset could be sold in an open market transaction. Similarly, as private equity fund managers have considerable discretion in their valuations, reported net asset values might suffer from a managed pricing phenomenon. This means that fund managers actively "manage" the pricing of their portfolios. In this sense, it is possible that fund managers mark the values of their portfolios up or down only when it is favorable to do so.

The impact of our paper to the existing literature is twofold. First, we present a new approach to measure the return of a private equity fund based on a stochastic

[1] See Chen et al. (2002) for references.

[2] The phenomenon of stale pricing in illiquid asset markets is extensively discussed in Getmanski et al. (2003) and Kaserer et al. (2003). For managed pricing in the private equity industry see Anson (2002).

model of a private equity fund's capital drawdowns and distributions over its lifetime. Applying equilibrium intertemporal asset pricing considerations, the stochastic model of capital drawdowns and distributions allows us to infer the fund's market value over time as the difference between the discounted sum of all outstanding future distribution and the discounted sum of all outstanding future capital drawdowns. This means, rather than using reported net asset values, we derive the market value of a fund based on the distribution of observable cash flows. This market value can then be used to define a periodic time-weighted return of a fund that does not suffer from indicated problems, such as stale or managed pricing. Second, our stochastic model of the dynamics of a private equity fund differentiates from the existing literature on private equity fund modeling in the following respects. The models of Takahashi and Alexander (2002) and Malherbe (2004) both rely on the specification of the dynamics of the unobservable value of the fund's assets over time, where model parameters have to be estimated from the disclosed net asset values of the fund management.[3] The dynamics of our model are solely based on oberservable cash flows, which seems to be a more promising stream for future research in the area of venture and private equity fund modeling.

The rest of the paper is organized as follows. Section 2 gives a definition of the financial market underlying our stochastic model of a private equity fund. Section 3 presents the stochastic model for the capital drawdowns of a fund over its lifetime. The stochastic model for the capital distribution is presented in Section 4. Section 5 shows how the market value and a periodic model time-weighted return of a fund can be calculated. Section 6 gives an conclusion and identifies areas for future research.

2 Definition of the Underlying Financial Market

We start with a precise definition of the underlying financial market, which will be relevant for our following stochastic model of a private equity fund. Consider a model of a financial market $\mathcal{S}$ that consists of one traded asset only: the risk-free asset or money market account.

Let r_f denote the deterministic short-rate of interest, which is assumed to be constant in the framework of our model. Under these assumptions, the dynamics of the money market account are given by the following ordinary differential equation:

$$dB_t = r_f B_t dt \tag{2.1}$$

We further assume that the price process of the money market account defined in (2.1) is normalized to 1, e.g. $B_0 = 1$.

The financial market $\mathcal{S}$ is modeled on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where the filtration $\mathbf{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions of saturatedness and right continuity. All Wiener processes z_t introduced throughout this paper are defined on and adapted to the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. That means, the probability space is large enough to support all random variables introduced hereafter.

[3] Note that Malherbe (2004) tries to account for the inaccurate valuation of the fund management by incorporating an estimation error in his model.

In addition, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ remain fixed.[4] From an economic perspective, this means that we do not incorporate important aspects of asymmetric distribution of information in the market for venture and private equity capital in our model.

We define a probability measure $\mathbb{Q}$ on $\mathcal{F}$ as being an equivalent local martingale measure to $\mathbb{P}$, if $\mathcal{S}$ is a local martingale under $\mathbb{Q}$.[5] If $\mathcal{M}^e(\mathcal{S})$ denotes the set of equivalent martingale probability measures, we define this set to be non-empty for all assets introduced hereafter:

$$\mathcal{M}^e(\mathcal{S}) \neq \emptyset \tag{2.2}$$

This condition is is sometimes called the "first fundamental theorem of asset pricing"[6] and is equivalent to the assumption of no-arbitrage on the financial market $\mathcal{S}$. We need this rather mild condition to ensure that all assets constructed in the following can be properly priced and we are able to deduce the market value of a private equity fund over its lifetime.

The model of the financial market $\mathcal{S}$ can be seen as the basis for our stochastic model of a private equity fund. We begin our model in the next Section with the description of the model of capital drawdowns of a fund over its lifetime.

3 A Stochastic Model of a Private Equity Fund's Capital Drawdowns

3.1 Assumptions of the Drawdown Model

The private equity fund to be modeled is a typical closed-end fund that is structured as a limited partnership. We differentiate between the fund's total (legal) maturity T_l and its commitment period T_c. The commitment period T_c denotes the time by which the general partners (GPs) of the fund can draw down capital from the investors or limited partners (LPs) of the fund. We assume that capital drawdowns of a fund occur unscheduled over the commitment period T_c, depending only on the investment decisions of the GPs. However, capital drawdowns over the whole commitment period can never exceed the total capital C, that was initially committed to the fund by the LPs. The total legal maturity of the fund is the time by which it is fully liquidated. Cumulated capital drawdowns from the *LPs* up to time t are denoted by D_t^c, undrawn committed capital up to time t by $\overline{D}_t$. From these assumptions the following simple relationships must hold:

$$D_t^c = C - \overline{D}_t, \quad \text{where} \quad D_0^c = 0 \quad \text{and} \quad \overline{D}_0 = C$$

Most empirical studies reveal that a funds capital distributions are heavily con-

[4]See Delbaen and Schachermayer (1997) p.2 for this assumption.
[5]See Schachermayer (2000).
[6]See Harrison and Kreps (1979), Harrison and Pliska (1981) and Delbaen and Schachermayer (1994).

centrated in the first few years of the funds lifetime.[7] We model this behaviour similar to the approaches of Takahashi and Alexander (2002) and Malherbe (2003), that assume capital to be drawn at a non-negative rate from the remaining undrawn committed capital.[8] Our assumption can be stated as follows:

Assumption 1 *Capital drawdowns over the commitment period T_c occur in continuous time. The behavior of the cumulated drawdowns D_t^c can be described by the ordinary differential equation (ODE)*

$$dD_t^c = \delta_t \overline{D}_t 1_{\{0 \leq t \leq T_c\}} dt, \tag{3.1}$$

where δ_t denotes the rate of contribution or simply the fund's drawdown rate at time t that is assumed to follow a non-negative stochastic process $\{\delta_t, 0 \leq t \leq T_c\}$.

The solution of this ordinary differential equation is well known from the bond pricing literature. Substituting the identity $dD_t^c = -d\overline{D}_t$ and using the initial condition $\overline{D}_0 = C$, yields the following solution of the ODE:

$$D_t^c = C - C \cdot \exp\left[-\int_0^t \delta_u du\right], \quad \text{where} \quad t \leq T_c \tag{3.2}$$

$$dD_t^c = \delta_t C \cdot \exp\left[-\int_0^t \delta_u du\right] 1_{\{0 \leq t \leq T_c\}} dt \tag{3.3}$$

The second equation gives the change of cumulated drawdowns between time t and $t + dt$, that is equal to the actual drawdowns that occur over an infinitesimally short time interval dt. As can easily be inferred from the second equation, the undrawn amounts up to time t exhibit exponential decay over the commitment period of the fund. Capital drawdowns are hence concentrated in the early years of a fund's lifetime under this framework. It is also worth to note, that this approach also incorporates the possibility that a fraction of the committed capital is not drawn from the LPs over the whole commitment period. A behavior that can often be observed in empirical studies.[9]

From equation (3.2) and (3.3), the conditional expected cumulated and instantaneous drawdowns can be inferred. If $E[\cdot|\mathcal{F}_s]$ denotes the expectations operator conditional on the filtration $\mathcal{F}_s$ defined in the preceding section, the expected cumulated drawdowns at time t are given by $(s \leq t)$:

[7] See Ljungqvist and Richardson (2003) for the US private equity market and Kaserer and Diller (2004) for the European market.

[8] Note that Takahashi and Alexander (2002) use a discrete-time framework to model drawdowns, whereas Malherbe (2004) assumes capital drawdowns to occur in a continuous-time framework similar to ours.

[9] For example, Ljungqvist and Richardson (2003) find that funds raised between the years 1981 and 1992 on average only invested 94.8 percent of their committed capital.

$$
\begin{aligned}
E[D_t^c|\mathcal{F}_s] &= E[C - \overline{D}_s \cdot \exp\left[-\int\limits_s^t \delta_u du\right] |\mathcal{F}_s] \\
&= C - \overline{D}_s \cdot E[\exp\left[-\int\limits_s^t \delta_u du\right] |\mathcal{F}_s] \qquad (3.4)
\end{aligned}
$$

The expected instantaneous drawdowns dD_t^c/dt are given by:

$$
\begin{aligned}
E[dD_t^c/dt|\mathcal{F}_s] &= \frac{d}{dt}E[D_t^c|\mathcal{F}_s] = \\
&= -\overline{D}_s \cdot \frac{d}{dt}E[\exp\left[-\int\limits_s^t \delta_u du\right] |\mathcal{F}_s] \qquad (3.5)
\end{aligned}
$$

By D_t^a we denote the actual drawdowns over a short time interval Δt, ranging from time t to $t + \Delta t$. The expected actual drawdowns $E[D_t^a|\mathcal{F}_s]$ over the time interval Δt can be approximated by:

$$
\begin{aligned}
E[D_t^a|\mathcal{F}_s] &= E[dD_t^c/dt \cdot \Delta t|\mathcal{F}_s] = \\
&= -\overline{D}_s \cdot \frac{d}{dt}E[\exp\left[-\int\limits_s^t \delta_u du\right] |\mathcal{F}_s] \cdot \Delta t \qquad (3.6)
\end{aligned}
$$

The conditional expectations in equations (3.4), (3.5) and (3.6) all require the integral over the stochastic process $\{\delta_t, 0 \leq t \leq T_c\}$ to be defined. In the next Section we will relate a fund's drawdown rate over time to the degree of competition on the market for private equity capital and derive a economically plausible stochastic process for the dynamics of the drawdown rate δ_t.

3.2 An Economic Model for the Dynamics of the Drawdown Rate

We start with a simple economic model for the market of private equity capital. In a similar fashion as the work of Inderst and Müller (2004), we define the market for private equity capital as being populated by entrepreneurial firms that search financing and private equity firms that provide financing for entrepreneurial projects.[10] On the market there is a finite set of entrepreneurial firms $\mathcal{M}_t^e$, where $|\mathcal{M}_t^e|$ denotes the total number of firms operating on the market at time t. The set of private equity firms on the market is given by $\mathcal{M}_t^p$, the corresponding number of private equity

[10]For an extensive description of the economics of the private equity market, see also Gompers and Lerner (1998).

firms at time t is given by $|\mathcal{M}_t^p|$. We explicitly added a time subscript t to the sets $\mathcal{M}_t^e$ and $\mathcal{M}_t^p$ to indicate that the number of entrepreneurial and private equity firms can vary over time. We assume that any private equity firm operating on the market $m_i^p \in \mathcal{M}_t^p$, for all $i \in \{1, ..., |\mathcal{M}_t^p|\}$, can finance entrepreneurial projects with capital that amounts to e_t^i. In a similar fashion, we assume that the financing requirements of an entrepreneurial firm $m_i^e \in \mathcal{M}_t^e$, for all $i \in \{1, ..., |\mathcal{M}_t^e|\}$, at time t is given by f_t^i. Under these assumptions, the total supply of private equity capital at time t is given by $S_t^p = \sum_i^{|\mathcal{M}_t^p|} e_t^i$. Equivalently, $D_t^e = \sum_i^{|\mathcal{M}_t^e|} f_t^i$ denotes the total demand for private equity capital of entrepreneurial firms operating on the market at time t. As a simple measure of the degree of competition on the market for private equity capital at time t, we define the ratio of total demand to total supply at time t, that is $Y_t \equiv D_t^e / S_t^p$. Low values of Y_t (values close to zero) indicate a low demand for private equity capital relative to the supply. That is in turn equal to a high degree of competition between private equity firms for attractive deals. On the other hand, high values of Y_t indicate that competition is low, as there is a relative shortage in capital supply compared to its demand.

We assume that the ratio Y_t is relatively stable over time. From time to time exogenous shocks on the capital supply or demand side increase or decrease the degree of capital market competition. Exogenous shocks on the demand side could, for example, be the invention of a new technology, like the development of the personal computer. If this exogenous shock is unpredictable, competition will fall after this shock because the stickiness of the private equity market leads to an initial shortage of private equity based capital. As time passes by, new private equity firms enter the market and the degree of competition reverts back to a more normal level.

Exogenous shocks on the capital supply side could, for example, be an improvement in the investment environment private equity firms operate in. Such an improvement in the investment environment leads to an increase in the number of private equity firms operating on the market and hence an increase in the degree of competition, which makes it harder for private equity firms to find attractive deals. Again, we assume that those shocks are only transitory and competition will revert back to a more normal level as time passes by.

In the following model we assume that the degree of capital market competition can be described by a single sufficient state variable. The assumptions we make are as follows:

Assumption 2 *The changes in the degree of capital market competition in the market for private equity capital Y_t can be described by a single state variable, Λ_t.*

That means that the state variable Λ_t can be thought of as determining the dynamics of the degree of competition Y_t in a way that the means and variances of Y_t are always proportional to Λ_t.

Assumption 3 *The dynamics of the state variable Λ_t can be described in continuous time by the stochastic differential equation (SDE)*[11]

$$d\Lambda_t = (\zeta + \xi\Lambda_t)dt + \nu\sqrt{\Lambda_t}dz_t, \tag{3.7}$$

where ζ, ξ and ν are constants, with $\zeta > 0$, $\xi < 0$ and $\nu > 0$.

[11] That is inspired by the work of Cox, Ingersoll and Ross (1985), that use a similar process to model the changes in production opportunities over time.

Under these assumptions, equation (3.7) corresponds to a mean-reverting process with a long-run mean of $-\zeta/\xi > 0$, a coefficient of mean-reversion $-\xi > 0$ and a volatility of $\nu > 0$. As the degree of capital market competition Y_t evolves proportional to Λ_t it will correspond to a mean-reverting process with a long-run mean proportional to $-\zeta/\xi > 0$. This is an economically reasonable behavior, as exogenous shocks to Y_t only have a transitory character and Y_t will always revert back to some long-run degree of competition.

In the next step, we relate the state variable Λ_t to the drawdown rate δ_t, at which a private equity fund draws down capital from its investors. From the theoretical work of Inderst and Müller (2004), it follows that a private equity fund's search time for new investments is an increasing function of the the ratio Y_t defined above. Hence the rate at which a private equity fund draws down capital will also increase with the degree of competition Y_t. Ljungqvist and Richardson (2003) test this proposition empirically and find a negative relationship between a fund's time to fully invested and the level of competition for deal flows. Motivated by these results, we make the assumption that the rate δ_t evolves proportionally to the state variable Λ_t.

Assumption 4 *The drawdown rate δ_t evolves proportional to the state variable Λ_t, that is*

$$\delta_t = c \cdot \Lambda_t, \tag{3.8}$$

where c is a constant, with $c > 0$.

From Itô's Lemma, it can be inferred that the drawdown rate δ_t follows a diffusion with drift and variance defined by:

$$Drift(\delta_t) = c(\zeta + \xi c\delta_t) = \kappa(\theta - \delta_t) \tag{3.9}$$

$$Var(\delta_t) = c^2\nu^2 c\delta_t = \sigma_\delta^2 \delta_t \tag{3.10}$$

Where κ, θ and σ_δ are positive constants. The dynamics of the drawdown rate δ_t can then be expressed by the stochastic differential equation:

$$d\delta_t = \kappa(\theta - \delta_t)dt + \sigma_\delta \sqrt{\delta_t} dz_t \tag{3.11}$$

This process is similar to the one proposed by Cox, Ingersoll and Ross (1985) for the short rate in their "Theory of the Term Structure of Interest Rates". As we have shown, the process is an economic plausible assumption for the dynamics of the drawdown rate. Furthermore, it has the advantage of mathematical tractability, as we will see in the following Section. In the next Section we will take advantage of Cox, Ingersoll and Ross (1985) and other related work to examine the behaviour of the drawdown rate under this spcification in some detail and derive a solution for the discounted sum of the outstanding drawdowns over a fund's lifetime.

3.3 Drawdown Rate and Drawdowns under the Specification of a Mean-Reverting Square-Root Process

We model the drawdown rate by a stochastic process $(\delta_t)_{t\in[0,T_c]}$, based on and adapted to the stochastic base $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ introduced above. The mathematical specification under the objective probability measure $\mathbb{P}$ is given by:

$$d\delta_t = \kappa(\theta - \delta_t)dt + \sigma_\delta\sqrt{\delta_t}dz_t \tag{3.12}$$

Where $\theta > 0$ is the long-run mean of the drawdown rate, $\kappa > 0$ governs the rate of reversion to this mean (coefficient of mean-reversion) and $\sigma_\delta > 0$ reflects the volatility of the drawdown rate. z_t is a standard Brownian Motion.[12] The specification of the drawdown rate as the square-root process defined in (3.12) has the advantage that it precludes negative values and is therefore an appropriate assumption. Furthermore, the mean-reverting structure of the process reflects the fact that we assume the drawdown rate to fluctuate randomly around some mean level θ.

As shown by Cox, Ingersoll and Ross (1985), the probability density function of the drawdown rate under this specification is the noncentral chi-square, $\chi^2[2c\delta_t; 2q+2, 2u]$, with $2q+2$ degrees of freedom and a paramter of noncentrality $2u$.[13]

The expected value and variance of the drawdown rate conditional on the filtration $\mathcal{F}_s$ are given by $(s \leq t)$:[14]

$$E[\delta_t|\mathcal{F}_s] = \delta_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) \tag{3.13}$$

$$Var[\delta_t|\mathcal{F}_s] = \delta_s\left(\frac{\sigma_\delta^2}{\kappa}\right)(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}) + \theta\left(\frac{\sigma_\delta^2}{\kappa}\right)(1 - e^{-\kappa(t-s)})^2 \tag{3.14}$$

If t grows, the expected drawdown rate converges to its long-run mean θ, as $\lim_{t\to\infty} E[\delta_t|\mathcal{F}_s] = \theta$. The variance of the drawdown rate converges to a constant positive number, as $\lim_{t\to\infty} Var[\delta_t|\mathcal{F}_s] = \theta\sigma_\delta^2/\kappa$.

Under the proposed dynamics of the drawdown rate, every contingent claim $\Phi(\delta_t, t)$ that is a function of δ_t and time t, including capital drawdowns, must satify the *fundamental differential equation*:[15][16]

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial\delta}[\kappa(\theta - \delta) - \lambda_\delta\sigma_\delta\delta] + \frac{1}{2}\sigma_{\delta^2}\frac{\partial^2\Phi}{\partial\delta^2} - r_f\Phi = 0 \tag{3.15}$$

Since the drawdown rate is not spanned by the assets in the economy, the drawdown risk can not be eliminated by arbitrage considerations.[17] Therefore, a

[12] Note that Wiener processes are also referred to as Brownian Motions in the literature.

[13] For details see Cox, Ingersoll and Ross (1985) p.391-392.

[14] See Cox, Ingersoll and Ross (1985) p.392.

[15] See Cox, Ingersoll and Ross (1985) p.393.

[16] For simplicity, the time subscript t is suppressed in the PDE.

[17] Remember from section 2, that we have assumed that the only traded asset on the financial market is the money market account.

market price of drawdown risk λ_δ is introduced that explicitly enters into the PDE above. This market price of risk is defined by:

$$\lambda_\delta \equiv \frac{E(d\delta_t) - r_f dt}{Std(d\delta_t)} = \frac{E(d\delta_t) - r_f dt}{\sigma_\delta \sqrt{\delta_t}} \tag{3.16}$$

From the derivatives literature it is well known, that in a complete financial markets setting, every contingent claim can be perfectly replicated by a portfolio of traded securities and thus pricing by arbitrage considerations alone is possible. When markets are complete, there exists a unique equivalent martingale measure (EMM) and the set of equivalent probability measures $\mathcal{M}^e(\mathcal{S})$ is therefore reduced to a singleton, $\{\mathbb{Q}\}$. This is also known in the literature as the "second fundamental theorem of asset pricing".[18]

In an incomplete markets setting, as considered here, pricing by arbitrage considerations is generally impossible, due to the existence of a non-empty set of different EMMs ($\mathcal{M}^e(\mathcal{S}) \neq \emptyset$), that yield different prices of contingent claims. Choosing one EMM out of the set $\mathcal{M}^e(\mathcal{S})$ is equivalent to specifying a certain functional form for the market price of risk λ_δ. In incomplete markets, prices are partly determined by aggregate supply and demand. Supply and demand for a certain security are in turn determined by the risk preferences of the investors on the market. If there were a traded asset on the market that pays $\delta(T)$ at time T, the price of this asset would already incorporate all the intertemporal equilibrium considerations associated with the pricing of drawdown risk and pricing by arbitrage would hence be possible. Under our setting of an incomplete market, we have to derive λ_δ from a general asset pricing model based on equilibrium considerations.

3.4 The Equilibrium Pricing of Drawdown Risk

To derive a formula for the market price of drawdown risk, we use the *consumption-based capital asset pricing model* (CCAPM), that goes back to the classic papers of Lucas (1978), Breeden (1979) and Grossman and Shiller (1981). We do not claim originality for this approach. Similar approaches can be found in the literature of option pricing under stochastic volatility. As volatility is a non-traded asset, the price of an option will also depend on the market price of volatility risk. To derive the functional form of the market price of volatility risk, for example, Heston (1993) uses the CCAPM.

From the consumption-based CAPM, it follows that the risk premium in (3.16) is defined by:[19]

$$E(d\delta_t) - r_f dt \equiv rra_t \cdot Cov(d\delta_t, dK_t/K_t) = rra_t \cdot \sigma_{\delta,K} \tag{3.17}$$

Where K_t denotes the consumption level of a representative investor at time t. rra_t is the relative risk aversion of the representative investor defined on its direct utility function for consumption $u(K_t)$, that is $rra_t = -K_t \cdot u''(K_t)/u'(K_t)$.

[18] See Dalbaen and Schachermayer (1994).

[19] See Breeden (1979) p. 275.

In the simplest case, the drawdown rate and aggregate consumption are uncorrelated ($\sigma_{\delta,K} = 0$). Under this specification, the market price of drawdown risk is equal to zero, $\lambda_\delta = 0$. In economic terms, that means that the drawdown rate carries no systematic risk and drawdown risk is hence not priced in the economy. Note that this is not an unrealistic case, as Ljungqvist and Richardson (2003) show that the rate at which private equity funds draw down capital is not correlated with conditions in the public equity markets.

For a non-zero correlation ($\sigma_{\delta,C} \neq 0$), the market price of risk λ_δ will depend on the coefficient of relative risk aversion. We assume that the representative investor has a power utiliy function of the form:

$$u(K_t) = \frac{K_t^{1-\gamma} - 1}{1-\gamma} \tag{3.18}$$

Where $\gamma > 0$ is the representative investor's coefficient of relative risk aversion. As a reference case, we can assume that the representative agent has log utility, that is $\gamma = 1$.[20] Under this assumption, the market price of risk is given by $\lambda_\delta = \sigma_K \rho_{\delta,K}$, where σ_K is the standard deviation of consumption and $\rho_{\delta,K}$ the correlation between the drawdown rate and consumption. The magnitude of λ_δ will then depend on whether the representative investor is more or less risk-averse than an investor with log utility. The specification of λ_δ will be as follows:

$$\begin{aligned} 0 < \gamma < 1 &\rightarrow \lambda_\delta < \sigma_K \rho_{\delta,K}, \\ \gamma = 1 &\rightarrow \lambda_\delta = \sigma_K \rho_{\delta,K}, \\ \gamma > 1 &\rightarrow \lambda_\delta > \sigma_K \rho_{\delta,K}. \end{aligned}$$

Using the results derived above, the stochastic process for the drawdown rate can be defined under the probability measure $\mathbb{Q} \in \mathcal{M}^e(\mathcal{S})$:

$$d\delta_t = [\kappa(\theta - \delta_t)dt - \lambda_\delta \sigma_\delta \sqrt{\delta_t}] + \sigma_\delta \sqrt{\delta_t} dz_t \tag{3.19}$$

In the following Section we use (3.19) to derive the expected cumulated and instantaneous drawdowns and find a formula for the discounted value of all outstanding drawdowns over the commitment period of a fund.

3.5 Expected Risk-Neutral Drawdowns and the Discounted Value of Outstanding Drawdowns

The expected cumulated drawdowns under the risk-neutral probability measure $\mathbb{Q}$ conditional on the information $\mathcal{F}_s$ revealed up to time s are given by ($s \leq t$):

[20] Note that as γ approaches one, the power utility function in (3.18) approaches the log utility function $u(K_t) = \ln K_t$.

$$
\begin{aligned}
E^{\mathbb{Q}}[D_t^c|\mathcal{F}_s] &= C - \overline{D}_s \cdot E[\exp\left[-\int\limits_s^t \delta_u du\right]|\mathcal{F}_s] \\
&= C - \overline{D}_s \cdot e^{A(s,t)-B(s,t)\delta_s}
\end{aligned} \tag{3.20}
$$

Where the functions $A(s,t)$ and $B(s,T)$ are well known from the work of Cox, Ingersoll and Ross (1985):[21]

$$
\begin{aligned}
A(s,t) &\equiv \frac{2\kappa\theta}{\sigma_\delta^2} \ln\left[\frac{2\alpha e^{[(\kappa+\lambda_\delta+\alpha)(t-s)]/2}}{(\kappa+\lambda_\delta+\alpha)(e^{\alpha(t-s)}-1)+2\alpha}\right] \\
B(s,t) &\equiv \frac{2(e^{\alpha(t-s)}-1)}{(\alpha+\kappa+\lambda_\delta)(e^{\alpha(t-s)}-1)+2\alpha} \\
\alpha &\equiv \left((\kappa+\lambda_\delta)^2+2\sigma_\delta^2\right)^{1/2}
\end{aligned}
$$

The expected instantaneous drawdowns are given by: [22]

$$
\begin{aligned}
E^{\mathbb{Q}}[dD_t^c/dt|\mathcal{F}_s] &= \frac{d}{dt}E^{\mathbb{Q}}[D_t^c|\mathcal{F}_s] = \\
&= -\overline{D}_s \cdot (A'(s,t) - B'(s,t)\delta_s)e^{A(s,t)-B(s,t)\delta_s}
\end{aligned} \tag{3.21}
$$

Where $A'(s,t) = \partial A(s,t)/\partial t$ and $B'(s,t) = \partial B(s,t)/\partial t$. Following the standard techniques from the derivatives literature, we can now value outstanding drawdowns at time t by the expectations of their risk-neutral discounted values, where expectations are computed with respect to the risk-neutral probability measure $\mathbb{Q}$. If we denote by $\underline{D}_t$ the discounted sum of all outstanding drawdowns at time t, it turns out:

$$
\begin{aligned}
\underline{D}_t &= E^{\mathbb{Q}}\left[\int_t^{T_c} e^{-r_f(\tau-t)}dD_\tau^c|\mathcal{F}_s\right] \\
&= -\overline{D}_s \cdot \int_t^{T_c} e^{-r_f(\tau-t)}(A'(s,t) - B'(s,t)\delta_s)e^{A(s,t)-B(s,t)\delta_s}d\tau
\end{aligned} \tag{3.22}
$$

Unfortunately, the integral in (3.22) cannot be eliminated and must hence be evaluated numerically. Now, we assume that drawdowns occur in discrete time and denote by D_t^a the actual drawdowns over a time interval Δt, ranging from time t to $t+\Delta t$. The discounted sum of all outstanding drawdowns $\underline{D}_t$ at time t can then be calculated by the discrete time equivalent to (3.22), that can be stated as:

[21] See Cox, Ingersoll and Ross (1985) p.393.

[22] See also equation (3.5).

$$
\begin{aligned}
\underline{D}_t &= E^{\mathbb{Q}}\left[\sum_{i=\frac{t}{\Delta t}+1}^{\frac{T_c}{\Delta t}} D_i^a e^{-r_f(i-\frac{t}{\Delta t})\Delta t}|\mathcal{F}_s\right] \\
&= \sum_{i=\frac{t}{\Delta t}+1}^{\frac{T_c}{\Delta t}} E^{\mathbb{Q}}\left[D_i^a|\mathcal{F}_s\right] e^{-r_f(i-\frac{t}{\Delta t})\Delta t}
\end{aligned}
\tag{3.23}
$$

Where $E^{\mathbb{Q}}\left[D_i^a|\mathcal{F}_s\right]$ can be approximated by:[23]

$$
E^{\mathbb{Q}}\left[D_i^a|\mathcal{F}_s\right] = -\overline{D}_s \cdot (A'(s, i\Delta t) - B'(s, i\Delta t)\delta_s)e^{A(s,i\Delta t)-B(s,i\Delta t)\delta_s}\Delta t \tag{3.24}
$$

Using (3.23) and (3.24), the value of the outstanding drawdowns over the fund's lifetime can be found. In the next Section, we present our model for the capital distributions of a private equity fund.

4 A Stochastic Model of a Private Equity Fund's Capital Distributions

4.1 Assumptions of the Distributions Model

In the following, we present our model for the capital distribution of a private equity fund. In our framework, capital distributions are defined as the positive cash outflows a fund distributes to its investors over its finite lifetime T_l. Our assumptions can be summarized as follows:

Assumption 5 *(Non-negative) capital distributions of a private equity fund to its investors occur between time t_0 and the fund's legal maturity T_l.*

We model distributions and drawdowns separately and therefore restrict distributions to be strictly non-negative at any time.

Assumption 6 *Capital distributions occur only at discrete times, $t_1 = t_0 + \Delta t$, $t_2 = t_0 + 2\Delta t$, ...,$t_N = t_0 + N\Delta t$, where $t_N = T_l$. Δt is defined to be the frequency at which the fund distributes capital to its investors and N is the total number of cash outflows, where $N = T_l/\Delta t$.*

As opposed to our model for the capital drawdowns of a fund, we do not assume that capital distributions occur in a continuous-time setting. However, as we will see in the following, the cash outflows in our framework will again be the result of some time-continuous stochastic processes.

Assumption 7 *Capital distributions at all discrete times t_n are assumed to follow a lognormal distribution, where $C_n \sim LN(\alpha_n, \beta_n)$ for all $n \in \{1, .., N\}$ holds.*

[23] See also equation (3.6)

The lognormal distribution has the advantage that it precludes capital distributions to become negative at any of the discrete times t_n and is therefore an economically reasonable assumption.

Assumption 8 *Furthermore, it is assumed that capital distributions at consecutive times are correlated with a coefficient of correlation* $\rho_{n,n+1}$ *for all* $n \in \{1, .., N-1\}$.

This is a simplifying assumption. We could easily extend the model to incorporate the correlation of capital distributions with higher lags. However, restricting the model to correlations between consecutive capital drawdowns makes it easier to implement in practice as less model parameters have to be estimated.

Figure 1 summarizes our assumptions. The value of all outstanding drawdowns discounted to time t_0, the funds starting date, is denoted by $\underline{D}_0$. The first capital distribution occurs at time t_0 and follows a lognormal distribution with distribution parameters of α_1 and β_1. This cash outflow is correlated with the capital distribution at time t_2, where the coefficient of correlation is $\rho_{1,2}$ and C_2 also exhibits the lognormal property with parameters α_2 and β_2. The last cash outflow of the fund occurs at time t_N, where t_N denotes the time by which the fund is fully liquidated.

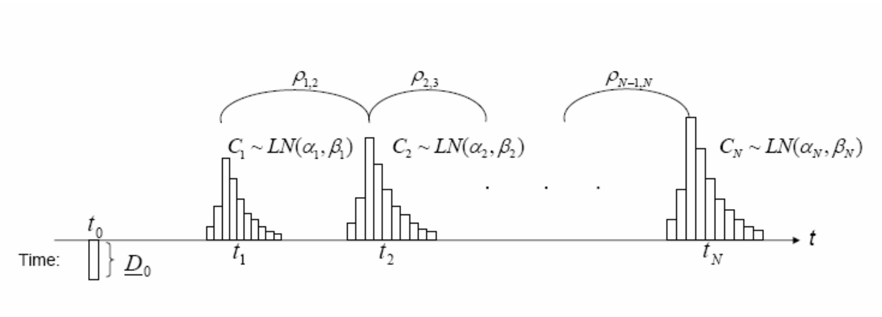

Figure 1: Model Assumptions of the Distributions Model

We explicitly use a discrete-time approach to model capital distributions because it seems to be more flexible than a continuous-time approach in dealing with the non-stationarities and serial correlation that can be found in empirical studies concerning the cash flows of private equity funds. For example, Ljungqvist and Richardson (2003) show that capital distributions of an average fund are not identically distributed over a fund's lifetime.[24] They report that capital distributions of an average fund are rare in the early years of a fund's lifetime and that most of the cash outflows are concentrated between year five and year 10. Kaserer and Diller (2004) report similar characteristics for European private equity funds.

In the next Section, we show how the capital distribution over the fund's lifetime can be modeled using hypothetical replicating assets.

[24]See Ljungqvist and Richardson (2003) table 3 on p.33.

4.2 Modeling Capital Distributions with Hypothetical Assets

To model the capital distributions of a fund, we define hypothetical assets that replicate the cash outflows at the discrete times times $t_1, ..., t_N$. The term hypothetical refers to the fact that these assets are only a construct that helps us to value the future cash flows and that we do not assume that these assets are traded in the underlying financial market. We keep the assumption of section 2 that the only traded asset in the market is the money market account. The economic idea behind our approach can be summarized as follows. At time t_0 when the fund is set up its net value must be equal to zero. Hence the discounted sum of all outstanding future distributions at t_0 must be equal to the discounted sum of all outstanding future drawdowns. The value of all discounted distributions must hence be equal to $\underline{D}_0$. We can consider this value to be the value of a portfolio of N hypothetical assets, that at time t_0 all have the same value of $\underline{D}_0/N$ and are defined to have the following properties.

Assumption 9 *An asset is defined to be the replicating asset to the cash flow of the fund at time t_n (for all $n \in \{1, ..., N\}$), if the random variable A_t^n modeling the price of asset n has the same distribution at time t_n as the cash flow C_n, that is $A_{t_n}^n \sim C_n$.*

From the lognormal distribution of the cash flows, it follows that the log cash flows must exhibit a normal distribution. Mathematically, this is:

$$\ln C_n \sim N(\alpha_n, \beta_n) \tag{4.1}$$

From the the distribution of the log cash flows, we can infer the distribution of the continuously compounded returns of the hypothetical assets. It follows, that for all $n \in \{1, ..., N\}$ it must hold:

$$\ln \frac{A_t^n}{A_0^n} = \ln C_n - \ln \frac{\underline{D}_0}{N} \sim N((\alpha_n - \ln \frac{\underline{D}_0}{N})/t_n, \beta_n/\sqrt{t_n}) = N(\mu_n, \sigma_n)^{25} \tag{4.2}$$

The continuously compounded returns for all assets hence follows a normal distribution, where μ_n denotes the expected return of asset n and σ_n its standard deviation. Under this specification, the dynamics of the prices A_t^n of the hypothetical assets can be formulated using simple geometric Brownian motions. The mathematical specification for all $n \in \{1, ..., N\}$ is:

$$\frac{dA_t^n}{A_t^n} = \mu_n dt + \sigma_n dz_t^n \tag{4.3}$$

Where z_t^n (for all $n \in \{1, ..., N\}$) are standard Wiener processes, for which $dz_t^n = \varepsilon_t^n \sqrt{dt}$ holds.[26] In Section 4.1 we have assumed that consecutive capital

[25]Note that μ_n and σ_n are measured as expected return and standard deviation of returns per year.

[26]Where ε_t^n are standard normal variates, where $\varepsilon_t^n \sim N(0, 1)$ holds.

distributions of a fund are correlated with a coefficient of correlation $\rho_{n,n+1}$. We can incorporate this correlation into our model by assuming that the Wiener processes of consecutive assets are correlated with $\rho_{n,n+1}$. That is:[27]

$$\rho_{n,n+1} = Corr\left(dz_t^n, dz_t^{n+1}\right)/dt \tag{4.4}$$

From the solution of the stochastic differential equation (4.3), it is possible to derive a closed-form formula for the cash flows that does only depend on the expected values and standard deviations of the log cash flows. The cash flows at each of the dates t_1,...,t_N are equal to the values of the corresponding assets at that time. That is for all $n \in \{1, ..., N\}$:

$$C_n = A_{t_n}^n = A_{t_0}^n \cdot \exp\left[\left(\mu_n - \frac{1}{2}\sigma_n^2\right)t_n + \sigma_n \varepsilon_n \sqrt{t_n}\right] \tag{4.5}$$

Substituting $A_0^n = \underline{D}_0/N$ and the expected returns and standard deviations μ_n and σ_n into this equation simplifies (4.5) to:

$$C_n = \exp\left[\alpha_n - \frac{1}{2}\beta_n^2 + \beta_n \varepsilon_n\right] \tag{4.6}$$

This equation can be used to find simulated values for the capital distributions for all of the $n \in \{1, ..., N\}$ cash outflows of a fund. In the next Section, we show how of the discounted sum of outstanding capital distributions can be found. As the hypothetical assets are not traded on a financial market, we must again apply equilibrium considerations.

4.3 Expected Risk-Neutral Distributions and the Discounted Value of Outstanding Distributions

Using a similar line of argument as in section 3.4, the risk-neutralized cash flows under the probability measure $\mathbb{Q}$ are defined as:

$$C_n = \exp\left[\alpha_n - \frac{1}{2}\beta_n^2 - \lambda_n \frac{\beta_n}{\sqrt{t_n}} t_n + \beta_n \varepsilon_n\right] \tag{4.7}$$

Where λ_n denotes the market price of risk of the hypothetical asset n, or conversely, the market price of risk of the nth cash flow of the fund. This market price of risk emerges because cash flows of the fund are constructed to depend on the values of the hypothetical assets that are not traded in the financial market. We further do not make the unrealistic assumption, that there exists "real" assets

[27] The correlation between the Wiener processes dz_t^n and dz_t^{n+1} can be realized by drawing the variates dz_t^n and dz_t^{n+1} from a standard bivariate normal distribution with correlation $\rho_{n,n+1}$. If x_1 and x_2 are independent standard normal variates, then the correlated variates can be found by setting $\varepsilon_t^n = x_1$ and $\varepsilon_t^{n+1} = \rho_{n,n+1}x_1 + x_2\sqrt{1-\rho_{n,n+1}^2}$.

in the market that are perfectly correlated with the defined hypothetical assets.[28] Hence, we are dealing with an incomplete market setting and pricing by arbitrage considerations is no longer feasible. Similar to Section 3.4, the market price of risk has to be inferred from equilibrium considerations. The market prices of risk λ_n are defined as:

$$\lambda_n \equiv \frac{E(dA_t^n/A_t^n) - r_f dt}{Std(dA_t^n/A_t^n)} \tag{4.8}$$

The risk-premium in the equation above can again be inferred from the consumption-based CAPM. It turms out:

$$E(dA_t^n/A_t^n) - r_f dt \equiv rra_t \cdot Cov(dA_t^n/A_t^n, dK_t/K_t) \tag{4.9}$$

Where rra_t is again the representative investor's coefficient of relative risk aversion and K_t denotes his consumption level at time t. Assuming a representative investor with power utility as in Section 3.4, it turns out that $rra_t = \gamma$. One might wonder whether the consumption-based CAPM is really applicable under this setting. However, it should be noted that Grossman and Shiller (1981) prove that the CCAPM does also hold for assets that are non-traded in financial markets.

The equation above requires as inputs the covariance between consumption changes and the returns of the hypothetical assets. These returns are not observable as the assets are only an artifical construct. The only observable quantities are the cash flows of the fund that occur at discrete times. We therefore approximate the risk-premium from above in discrete time by defining:

$$E(\Delta A_t^n/A_{t-1}^n) - r_f \approx \gamma \cdot Cov(\ln C_{t_n}, \ln K_{t_n}) \cdot 1/t_n = \gamma\sigma_{C,K}/t_n \tag{4.10}$$

Where $\sigma_{C,K}$ is the covariance between the log cash flows of the fund at time t_n and the log consumption at that time.[29] Substituting these results into the equation for the cash flows gives:

$$C_n = \exp\left[\alpha_n - \frac{1}{2}\beta_n^2 - \gamma\sigma_{C,K} + \beta_n\varepsilon_n\right] \tag{4.11}$$

The conditional expectations of the capital distributions under the probability measure $\mathbb{Q}$ are then given by:[30]

$$E^{\mathbb{Q}}[C_n|\mathcal{F}_s] = \exp\left[\alpha_n - \frac{1}{2}\beta_n^2 - \gamma\sigma_{C,K}\right] \tag{4.12}$$

[28] Respectively, perfectly correlated with the cash flows of the fund.

[29] Note that we make the simplifying assumption that this covariance is identical for all cash flows.

[30] Note that the conditional expectations are equal to the unconditional expectations, as we have assumed α_n and β_n to be given exogenously.

Using standard arguments from the derivatives literature, the discounted value of all outstanding capital distributions at time $t \in \{t_0, ..., t_N\}$ can then be calculated by:

$$\begin{aligned}\underline{C}_t &= E^{\mathbb{Q}}[\sum_{i=\frac{t}{\Delta t}+1}^{N} C_i \cdot e^{-r_f\left(i-\frac{t}{\Delta t}\right)\cdot\Delta t}|\mathcal{F}_s] = \\ &= \sum_{i=\frac{t}{\Delta t}+1}^{N} E^{\mathbb{Q}}[C_i|\mathcal{F}_s] \cdot e^{-r_f\left(i-\frac{t}{\Delta t}\right)\cdot\Delta t} \end{aligned} \tag{4.13}$$

Where $E^{\mathbb{Q}}[C_i|\mathcal{F}_s]$ is given by equation (4.12). In the next Section, we can now define the market value and model retun of a private equity fund.

5 Market Value and Model Time-Weighted Return of a Private Equity Fund

Using the results from the stochastic model of capital drawdowns and distributions of a private equity fund, we can now infer a formula for the fund's market value MV_t at time t. If $\underline{C}_t$ and $\underline{D}_t^a$ denote the discounted sums of all outstanding distributions and drawdowns at time t, the market value MV_t is given by:

$$MV_t = \underline{C}_t - \underline{D}_t^a + E^{\mathbb{Q}}\left[MV_{T_l} \cdot e^{-r_f(T_l-t)}\right] \tag{5.1}$$

Where $E^{\mathbb{Q}}\left[MV_{T_l} \cdot e^{-r_f(T_l-t)}\right]$ is the funds risk-neutral market value at liquidation T_l discounted to time t. As private equity funds are fully liquidated at the end of their lifetime T_l, this value is per definition equal to zero. The fund's market value is then completely defined by the difference between the discounted sum of all outstanding capital distributions and drawdowns:

$$MV_t = \underline{C}_t - \underline{D}_t^a \tag{5.2}$$

In discrete-time, when drawdowns and distributions occur with a frequency of Δt, the market value MV_t at time $t \in \{t_0, ..., t_N\}$ can be written as:

$$\begin{aligned}MV_t &= E^{\mathbb{Q}}[\sum_{i=\frac{t}{\Delta t}+1}^{N} C_i \cdot e^{-r_f\left(i-\frac{t}{\Delta t}\right)\cdot\Delta t}|\mathcal{F}_s] - E^{\mathbb{Q}}[\sum_{i=\frac{t}{\Delta t}+1}^{N} D_i^a \cdot e^{-r_f\left(i-\frac{t}{\Delta t}\right)\cdot\Delta t}|\mathcal{F}_s] = \\ &= \sum_{i=\frac{t}{\Delta t}+1}^{N} E^{\mathbb{Q}}[C_i|\mathcal{F}_s] \cdot e^{-r_f\left(i-\frac{t}{\Delta t}\right)\cdot\Delta t} - \sum_{i=\frac{t}{\Delta t}+1}^{N} E^{\mathbb{Q}}[D_i^a|\mathcal{F}_s] \cdot e^{-r_f\left(i-\frac{t}{\Delta t}\right)\cdot\Delta t} \end{aligned} \tag{5.3}$$

Where $E^{\mathbb{Q}}[C_i|\mathcal{F}_s]$ and $E^{\mathbb{Q}}[D_i^a|\mathcal{F}_s]$ are given by equations (4.12), (3.23) and (3.24) above. With this market value, a periodic time-weighted return of a fund over a time interval Δt can be defined by:

$$r^d_{t,t+\Delta t} = \frac{MV_{t+\Delta t} + C_{t+\Delta t} - D^a_{t+\Delta t} - MV_t}{MV_t} = \\ = \frac{MV_{t+\Delta t} + C_{t+\Delta t} - D^a_{t+\Delta t}}{MV_t} - 1 \tag{5.4}$$

Or in terms of continuously compounded returns:

$$r^s_{t,t+\Delta t} = \ln\left[\frac{MV_{t+\Delta t} + C_{t+\Delta t} - D^a_{t+\Delta t}}{MV_t}\right] \tag{5.5}$$

These returns have all the desirable properties of standard time-weighted returns used is portfolio and risk management models. They allow the estimation of a standard deviation of returns and a correlation of private equity returns to other asset classes. The returns do only depend on the defined market values of a fund and the observable capital distributions $C_{t+\Delta t}$ and drawdowns $D^a_{t+\Delta t}$ over the time interval Δt. They do not depend on net asset values reported by the fund management and do therefore not suffer from problems, such as stale or managed pricing. Furthermore, this approach allows us to perform a Monte-Carlo simulation to examine the distribution of the returns on a private equity fund under different parameter specifications.

6 Conclusion

In this paper, we present a stochastic model for the dynamics of a private equity fund. Our work differentiates from previous research in the area of venture and private equity fund modeling in the following respects. Our model of a fund's capital drawdowns and distributions is based on observable economic variables only. In this sense, we do not specify a process for the dynamics of the unobservable value of a fund's assets over time, as done in the existing deterministic and stochastic models of Takahashi and Alexander (2002) and Malherbe (2004). Rather, we endogenously derive the market value of a fund by using equilibrium intertemporal asset pricing considerations. The combination of equilibrium asset pricing principles and appropriate economic modeling of the underlying stochastic processes allows us derive a simple closed-form solution for the market value of a fund over time. This market value can be used to define a periodic model return of a private equity fund. Hence, we do also provide a new approach to measure the return of private equity investments that does not suffer from typical problems, such as stale or managed pricing.

To our knowledge, we are the first that apply considerations along these lines. In addition, the methods developed here could easily be applied to other illiquid alternative investments, such as closed-end real estate funds. Another stream of research would be the extension of our model to more complex stochastic settings.

Furthermore, an empirical test and calibration of our model to empirical data is left for future research.

References

Anson, M. J. P. (2002). Managed Pricing and and the Rule of Conservatism in Private Equity Portfolios, Journal of Private Equity, Spring 2002.

Breeden, D. T. (1979). An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities, Journal of Financial Economics, 7, 265-296.

Chen, P., Baierl, G. T., Kaplan, P. D. (2002). Venture Capital and its Role in Strategic Asset Allocation, Journal of Portfolio Management, Winter 2002, 83-89.

Cox, J. C., Ingersoll, J. E. and Ross, S. A. (1985). A Theory of the Term Structure of Interest Rates, Econometrica, Volume 53, Issue 2, 385-408.

Delbaen, F. and Schachermayer, W. (1994). A General Version of the Fundamental Theorem of Asset Pricing, Mathematische Annalen, Volume 300, 463-520.

Delbaen, F. and Schachermayer, W. (1997). Non-Arbitrage and the Fundamental Theorem of Asset Pricing: Summary of Main Results, Proceedings of Symposia in Applied Mathematics, Volume 60, 1-10.

Harrison, J. M. and Kreps, D. (1979). Martingales and Arbitrage in Multiperiod Securities Markets, Journal of Economic Theory, Volume 20, 381-408.

Getmanski, M., Lo, A. W. and Makarov, I. (2003). An Econometric Model of Serial Correlation and Illiquidity in Hedge Fund Returns, MIT Working Paper.

Gompers, P. A. and Lerner, J. (1998). What Drives Fundraising?, Brooking Papers on Economic Activity: Microeconomics, 149-204.

Grossman, S. J. and Shiller, R. J. (1981). Consumption Correlation and Risk Measurement in Economies with Non-Traded Assets and Heterogeneous Information, NBER Working Paper No. 690.

Harrison, J. M. and Pliska, S. R. (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading, Stochastic Processes and their Applications, Volume 11, 215-260.

Heston, S. L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, Review of Financial Studies, 6(2), 327-343.

Hirshleifer, Jack (1970). Investment, Interest and Capital. Englewood Cliffs, NJ: Prentice-Hall.

Inderst, R. and Müller, H. M. (2004). The Effect of Capital Market Characteristics on the Value of Start-Up Firms, Journal od Financial Economics, 72, 319-356.

Kaserer, C. and Diller, C. (2004). European Private Equity Funds - A Cash Flow based Performance Analysis, CEFS Working Paper.

Kaserer, C., Wagner N. and Achleitner A. (2003). Managing Investment Risks of Institutional Private Equity Investments - The Challenge of Illiquidity, CEFS Working Paper.

Ljungqvist, A. and Richardson, M. (2003). The Cash Flow, Return and Risk Characteristics of Private Equity, NBER Working Paper 9454.

Lucas, R. E. (1978). Asset Prices in an Exchange Economy, Econometrica 46, 1429-1445.

Malherbe, E. (2004). Modeling Private Equity Funds and Private Equity Collateralized Fund Obligations, International Journal of Theoretical and Applied Finance, Volume 7, No. 3, 193-230.

Schachermayer, W. (2000). Introduction to the Mathematics of Financial Markets, Working Paper.

Takahashi, D. and Alexander, A. (2002). Illiquid Alternative Asset Fund Modeling, Journal of Portfolio Management, Winter 2003, 90-100.