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Keywords: asset-liability-management (ALM), internal risk model, portfolio optimization, risk capital

JEL-classification: G11, G22, G32

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A. Introduction

The recent development in financial markets calls for stronger regulatory requirements of financial intermediaries focusing not only on banks but also on the insurance sector, where important aspects are an all encompassing control of business risk as well as the discussion regarding the required amount of capital necessary to cover for the risk. A major challenge – for regulators as well as risk managers – when dealing with risk is the operationalization in general and the mapping of a company's particular risk portfolio onto an opportunities-threats matrix to capture the overall risk including the worst-case scenario cash outflows, i.e. the development of an all encompassing risk model. The topic of risk modeling is also one of the focal points of the European Solvency II-project, where the usage of internal risk models to determine the capital requirements has steadily increased in relevance (IAA, 2004). In addition to the solvency regulation, recent modifications to the international financial reporting standards have resulted in consequences for determining the necessary risk capital (Bloomer, 2005; Meyer, 2005), further increasing the relevance of risk modeling to become one of the central issues in the insurance industry.

Beside the external regulatory requirements, internal managerial aspects motivate a risk and capital monitoring management as well. Here the three traditional goals for managing insurance companies serve as motivators: Maximizing profits (often proposed as the main goal) and maximizing growth whilst keeping risk at an acceptable level to avoid insolvency (which is also required by regulators), i.e. safety first as popularized by Cramér (1930) for the theory of risk in insurance companies, Roy (1952) for holding assets.[1] The later objective is of particular importance in the insurance market, where the ruin probability is of great consequence given that a direct relationship between the probability to default and the perceived quality of the offered products seems to exist (Cummins & Sommer, 1996; Wakker, Thaler, & Tversky, 1997). As a result, adequate management tools serving the objective of maximizing profit and growth and simultaneously keeping risk at an acceptable level are required.

In general, the main factors relevant for the ruin probability are the risks generated through operations, i.e. the risks originating from asset-management/allocation as well as the risks originating from insurance policies, referred to as insurance risk. We define insurance risk to be the risk of incurring an insurance claim that exceeds the insurance premium in a given period. Generally stated, the higher the underwriting risk an insurer assumes the more constrained he is in assuming investment risk in the asset portfolio, where the amount of underwriting risk seems the determining factor given that the business model of an insurer involves the provision of insurance coverage. Maintaining a particular level of overall ruin probability for any level of operating risk is achieved through the allotment of risk-based capital, in the following referred to as risk capital, where we use the term risk capital in the tradition of Butsic (1994), who defines risk capital in general to be the theoretical amount of capital needed to absorb the risks of conducting a business with a predefined level of certainty. For insurers in particular the level of risk capital is the amount of capital that covers exceptional losses, where exceptional loss is defined as the loss that exceeds the sum of the paid in premiums in combination with the gains accumulated from the asset investments. The required level of risk capital is therefore a function of an acceptable ruin probability, the level of the insurance premiums, the risk of the overall position of the insurance contracts given a certain time horizon, and the asset allocation decision of the insurer.

1 More recent approaches subsume the three goals under value based management with the primary objective to maximizing shareholder wealth.

For the individual insurer determining the necessary periodic level of risk capital is model-dependent and regularly operationalized using a one-period ruin probability – often either set to a subjective level or corresponding to the probability to default of a given rating – and a model incorporating the insurance risk, the asset portfolio risk, and the risk-aversion of the market participants, i.e. the insurees as well as the equity holders of the insurer. Recently – under the new solvency rules – regulators have expressed a preference for internal risk management models to determine the level of risk capital. Here tail value at risk measures, such as the value at risk (VaR) or the conditional value at risk ($CVaR$), recommended as appropriate risk measure to determine the necessary amount of risk capital by the IAA (2004), are of increasing popularity (Eling & Parnitzke, 2007; Eling, Schmeiser, & Schmit, 2007; Tasche, 2002; von Bomhard, 2005). Evidence of the usefulness of regulatory capital requirements for financial institutions based on the VaR is presented by Cuoco and Liu (2006) who find that VaR based capital requirements can be very effective not only in limiting portfolio risk but also in inducing financial institutions to reveal the risk of their investments and to adequately support this risk with capital.

We decide to utilize the $CVaR$ as risk measure when deriving an insurer's surplus whilst simultaneously capturing the dependencies within and between the asset and liability portfolios, due to its property of being a coherent risk measure[2] for normal as well as for non-normal distributed risks. For evaluations of the quality of the VaR or $CVaR$ as risk measure refer to Artzner et al. (1997; , 1999), Acerbi and Tasche (2002), Alexander and Baptista (2004), Alexander et al. (2007), Tasche (2002), or Yamai and Yoshiba (2005), to name but a few. For a comprehensive overview on alternative quantile based risk measures to the VaR refer to Dowd and Blake (2006) and for risk capital allocation using coherent risk measures based on one-sided moments of very general distributions refer to Fischer (2003).

In general, when optimizing portfolios the utilization of tail value at risk measures leads to solving a stochastic optimization problem where multiple algorithms have been proposed (Birge & Louveaux, 2000; Ermoliev & Wets, 1988; Kall & Wallace, 1995).[3] The utilization of tail value at risk measures in portfolio management has been demonstrated by for instance Erik, Romeijn, and Uryasev (2001) who apply the $CVaR$ to a ALM problem of pension funds, or by Krokhmal et al. (2002). Ferstl and Weissensteiner (2009) similarly apply a multi-stage stochastic linear programming ALM to a pension scheme using a $CVaR$ framework and Rockafellar and Uryasev (2000) detail an approach to optimize a portfolio whilst minimizing the VaR and $CVaR$ simultaneously. An evaluation of the fundamental properties of the $CVaR$ in the context of loss

2 With coherence being defined according to the axioms of Artzner et al. (1997; , 1999).
3 For particular applications in finance refer to Ziemba and Mulvey (1998) or Zenios (1996).

distributions has been provided by Rockafellar and Uryasev (2002) and a practical application of credit risk mitigation is for instance detailed by Bucay and Rosen (1999) or Andersson et al. (2001).

By implementing ALM in a context of absolute values and in establishing a direct relationship to the required risk capital in the optimization procedure, we extend the works of Butsic (1994), who has detailed the various implications of dependencies amongst assets and liabilities of insurance companies in the context of ruin probability and risk capital. Further, we expand the model by Li and Huang (1996), who determine the optimal composition of the insurance and investment portfolio of a property-liability insurance company constrained by a risk-threshold level to be defined by the insurer, by implementing a tail value at risk measure and establishing a relationship to the required risk capital. Our works is also related to the model of Cummins and Nye (1981), who apply mean-variance efficient portfolio theory to determine the efficient frontier, where when applying the safety-first criterion the ruin probability is a consequence of the selected product mix, i.e. the position on the efficient frontier. We extend their model by illustrating the dependency of the level of risk capital on the asset allocation and show that an optimal asset allocation exists that maximizes the return on risk capital whilst utilizing the diversification effects of an ALM optimization. We include non-linear dependencies using alternative claim distributions in our analysis and show that optimizing the asset allocation and thereby defining the level of risk capital to meet the regulatory targets is of importance, not only to satisfy the statutory requirements but also from a shareholder perspective.[4] We thereby elaborate on the research of Haugen and Kroncke (1970), Scheel et al. (1972), and Cummins and Nye (1981) who mention in a side note that this optimization technique might be used to maximize the value of equity.

Overall, we contribute to the existing literature by showing that an ALM optimization with direct inclusion of risk capital in dependence on the total portfolio risk leads to an overall optimum of the firm's cash-flow and risk situation. I.e. the additional diversification effect of ALM in comparison to pure asset-management allows for a reduction of required risk capital and the maximization of return on capital. Furthermore, we show that for constrained efficient markets – such conditions as provided by the regulatory requirements in the insurance industry – risk management adds value to shareholders. We generalize our analytical findings for normal

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[4] A model and the consequent management of alternative loss distributions consisting of non-catastrophe and catastrophe losses with non-linear dependencies has been previously investigated by Eling and Toplek (2009) based on the model presented by Eling et al. (2008). We take a different approach by illustrating the effect of various distributions on the risk capital and propose an optimal ALM solution to maximize the return on capital considering a tail value at risk measure and establishing a relationship to the required risk capital.

distributed claims to different claim distributions using Monte Carlo simulation. The paper is structured as follows. First, we integrate our research in the existing literature. Second, we derive our model analytically, and third we generalize our findings to different claim distributions. A discussion of the implications and further research concludes.

B. Asset-liability-management applied to an insurance context

Asset-liability-management (ALM), which in general is assigned a particular importance in the insurance industry (Cummins & Nye, 1981; Kahane & Nye, 1975), allows for a systematic management of risks induced through the assets as well as the liabilities of the insurer. ALM in this context mainly serves the achievement of financial stability and thereby affects the quality of the products, guarantees the compliance with regulatory requirements, and grants a risk adequate return for equity holders.[5] We focus on the particular aspect of managing assets and liabilities in a portfolio context and view insurance companies as levered financial institutions holding financial assets to ascertain the coverage of liabilities generated through the underwriting activity. This allows for a separate evaluation of two distinct portfolios held by the insurer. On the liability-side the insurer holds a portfolio consisting of different insurance lines, such as auto, fire, etc., generating premium income and claims. The pool of funds available for investment – on the asset-side of the insurer - consists of the paid in premiums as well as of the risk capital supplied by the owners.

It is reasonable to assume that both portfolios are risky and not independent. This proposition is supported by empirical evidence regarding the correlation coefficients between assets and liabilities (Butsic, 1994; Kahane & Nye, 1975), the correlation coefficients between insurance lines and mutual funds (Haugen, 1971), the correlation coefficients between insurance profits and stock market returns (Biger & Kahane, 1978), as well as the correlations between the various insurance lines and the dependencies between loss and expense ratios for various lines of property and liability insurance (Lambert Jr & Hofflander, 1966). For more recent data on estimates for a property-liability insurance company refer to Li and Huang (1996) alternatively for a plausible argument for the utilized correlation structure refer to the argument in Eling et al. (2008, pp. 664-665). The stochastic nature of the asset and insurance portfolio in combination with the existence of dependencies amongst the various positions allows for ALM according to modern portfolio theory, where managing assets entails the utilization of diversification effects through efficient asset allocation and managing liabilities offers additional diversification potential

5 For a comprehensive overview on various measures mentioned in the context of ALM refer to Lamm-Tennant (1989).

through utilizing the not perfect correlations of the claim distributions of various insurance lines. Combining both effects results in an efficient portfolio for the insurer.

Utilizing this kind of risk-return optimization of the overall position of the composite insurer is not an entirely new idea and was initially proposed by Michaelsen and Goshay (1967), Ferrari (1967), and Haugen and Kroncke (1970), the later are subject to an important comment by Scheel et al. (1972) clarifying some important issues regarding the relevance of dependencies amongst the stochastic variables.[6] Various alternative ALM optimization models were consequentially developed, such as the models proposed by Kahane and Nye (1975), Quirin and Waters (1975), Markle and Hofflander (1976), Kahane (1977a; , 1977b; , 1977c), Cummins and Nye (1981), Butsic (1994) and Dus and Maurer (2001). Most recently simulative approaches not relevant for our analytical model have increased in popularity, such as multistage stochastic programming models to determine the optimal asset allocation strategy integrating equity exposure as a function of risk capital (Carino et al., 1994), dynamical models utilized to simulate cash flows in order to forecast assets, liabilities, and ruin probabilities for different scenarios (Eling & Parnitzke, 2007), or the simulation of ALM utilizing a discrete model (Gerstner, Griebel, Holtz, Goschnick, & Haep, 2008). Overall, it is well established that ignoring the dependencies of assets and liabilities results in inefficient positions (Krouse, 1970; Pyle, 1971).

When applying the above mentioned models the resulting optimization most often reduces the activity of the insurer to a few insurance lines only, implying the abandonment of the other lines. Whereas, the asset portfolio in general is easily restructured, the modification of the activity in existing insurance lines seems problematic. Arguments against an unconstrained modification of the liability side are often based on marketing, market share, and cross-selling arguments; the most prominent argument being that for competitive reasons most insurers must offer a full range of coverage. A similar argument is posed by product complementarities, where, for instance, in most cases auto liability and auto physical damage insurance are written as a package. The financial results from the two insurance lines, however, are evaluated separately, where the profitability of the two lines often differs. If the company decides that one of the lines is undesirable, the cutbacks in that line will be complemented with cannibalization effects in the other line. Furthermore, the costs associated with establishing any given line of business rules a spontaneous withdrawal from any particular line except under the most severe circumstances out. An abrupt increase in premium volume in any given line could lead to a compromising of the company's underwriting capacity and would most likely lead to the acceptance of substandard

6 It should be noted that for empirical testing of their propositions Haugen (1971) did not develop the entire efficiency frontier and Ferrari (1967) developed only the efficient insurance frontier.

business. In conclusion, an unrestricted ceteris paribus optimization of the contracting volume of the various insurance lines is not realistic.

An improvement of this unrealistic optimization solution is a limitation of the possible ranges of weights for the insurance lines during the optimization process, as has been proposed by various authors (Cummins & Nye, 1981; Ferrari, 1967; Kahane & Nye, 1975). Although improving the unconstrained models we are still of the opinion that adapting the activity in various insurance lines to the optimal ALM solution is rarely possible in the short-run. Granted that it is possible to restructure the insurance portfolio with some effort and financial commitment over a longer period of time, it is still questionable whether the optimal structure at a particular point in time is stable over time and still optimal at a later point. Considering the dependence of the optimal solution on the correlation structure amongst assets, liabilities, and the market, the existence of an inter-temporal optimum seems not plausible.

Additional criticism can be expressed regarding the assumption – common to all models – that the return on premium income is independent of the quantity of the insurance contracts. This proposition assumes that the pricing of insurance premiums is entirely market dependent, i.e. the market pays only the actuarially fair premium, which disregards the possibility of an active pricing policy of the insurer. Also, economies of scale as well as pooling effects due to higher insurance volumes are ignored. Finally, all of the reported studies utilize relative measures, i.e. returns of the asset portfolio and returns of the insurance portfolio. This approach has recently been criticized, since evaluating risk utilizing relative measures is commonly questioned in discussions regarding solvability and Solvency II. Here the concurrent opinion is that measurements utilizing an absolute value are preferred (IAA, 2004). Ruin probability, if included in the models, serves only as constraint truncating the efficient frontier.

We, in the following model the optimal ALM-mix for a 1-period insurance portfolio and an asset portfolio utilizing absolute values when optimizing the portfolios to allow for the inclusion of an absolute risk measure in the optimization process.

C. Model

We initially illustrate the expected effects of ALM using normal distributed claims in Figure 1. In the subsequent section the model is formulated for normal distributed asset returns and claims and a linear dependency structure and extended to include the regulatory constraint of a particular ruin probability. Figure 1 illustrates the claim distribution – before and after assessing the full optimization potential – with the expected claim denoted as $E(C)$. The aggregated premiums (π) and the expected cash flows from the asset portfolio ($E(R)$), in total $\pi + E(R)$, denote the capital provide to cover the claims. The remaining gap to obtain coverage of $1-\varepsilon$ is filled by risk capital (RC_ε), where the 1-period ruin probability ε is defined as the probability that the claims of one period do not exceed the cash flow plus the risk capital of that period, i.e. $P(C > \pi + RC + R) \leq \varepsilon$. For an optimized ALM solution, i.e. a decrease of the overall variance ($\sigma_{CF,ALM}$), the confidence level of $1-\varepsilon$ is reached with a lower risk capital ($RC_{\varepsilon,ALM}$) requirement. For a successful reduction of the risk the absolute level of risk capital is reduced whilst maintaining the same level of premiums and expected return on the asset portfolio.

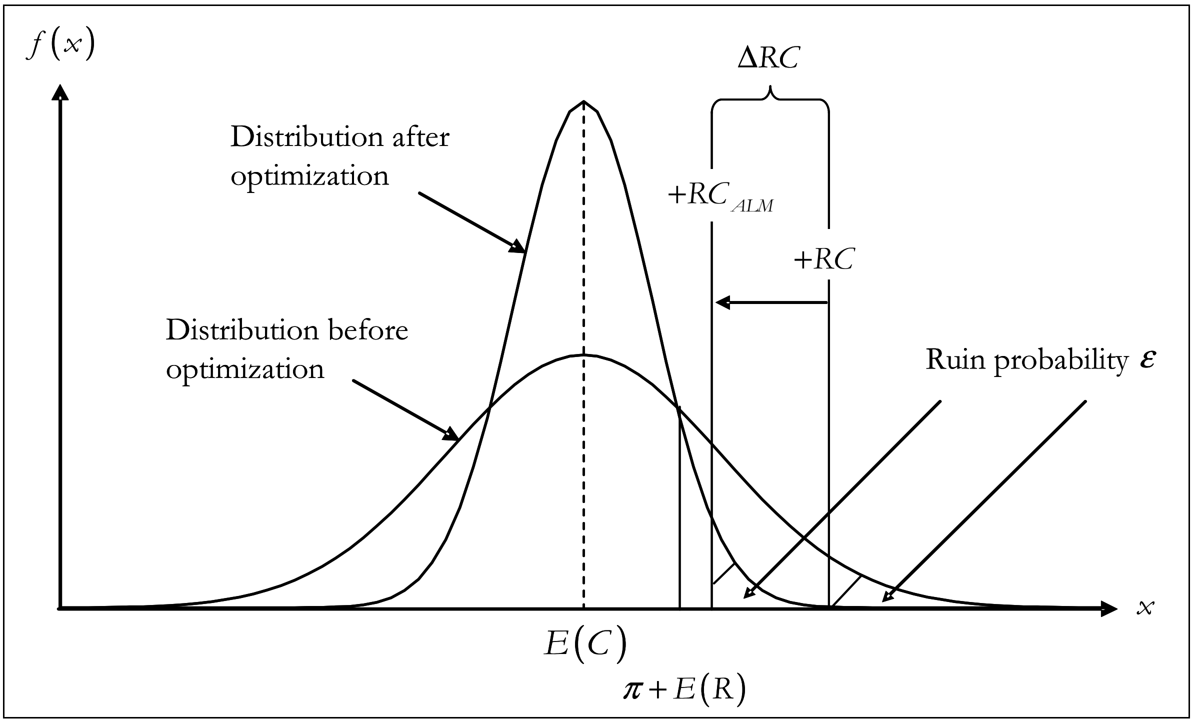

Figure 1: Illustration of the effect of an ALM optimization on the required risk capital depicted as net claim distribution.

As a consequence of the decrease in risk capital the return on the required risk capital for a particular ruin probability is increased, where the expected return on risk capital ($E(r_{RC})$) is $E(r_{RC}) = E(CF)/RC_\varepsilon - 1$, with $E(CF)$ being defined as $E(R) + \pi - E(C)$. Given that for most reasonable dependence structures of the asset and insurance portfolio a diversification potential through ALM exists, relation $\sigma_{CF,ALM} < \sigma_{CF}$ holds. The difference between RC_ε and $RC_{\varepsilon,ALM}$ is ceteris paribus determined according to (1), where $N_{1-\varepsilon}$ is denoting the $(1-\varepsilon)$-quantile of the normal distribution identifying a certain ruin probability and φ the density function of the standard normal distribution.[7] The reduction in risk capital consequently leads to an increase in return on risk capital according to (2).

$$RC_{\varepsilon} - RC_{\varepsilon,ALM} = \frac{\varphi(N_{1-\varepsilon})}{\varepsilon} \cdot \left(\sigma_{CF} - \sigma_{CF,ALM}\right) \tag{1}$$

$$E\left(r_{RC,ALM}\right) = \frac{E(CF)}{RC_{\varepsilon,ALM}} - 1 > E\left(r_{RC}\right) = \frac{E(CF)}{RC_{\varepsilon}} - 1 \tag{2}$$

This result is not surprising given the findings from modern portfolio theory, where for correlated asset classes – irrespective whether they consist of assets, claims, or a combination of both - diversification potential exists. We in the next section analytically implement a model relaxing the ceteris paribus constraint. Moreover, we investigate – with a direct focus on the minimum required risk capital – whether an optimal combination of assets and liabilities can be achieved.

The model is derived analytically for the simplified state of normal distributed asset returns and claims and a linear dependency structure between the asset return and the claim distributions. Later on, we generalize the findings to various claim distributions and non-linear relations utilizing Monte Carlo Simulation. For the model we assume that individuals are risk-avers and apply the mean-variance criterion in selecting their portfolios.[8] Markets are efficient, i.e. transaction costs and taxes do not exist, and information is costless. The optimization model is illustrated for a P/L insurer that holds a portfolio of insurance policies (from different insurance lines) and has invested the proceeds from the contracts in various asset classes. Insurance contracts mature in one period, the premiums are paid at the beginning of the period and are available for investment in assets, where one dollar of premium in any insurance line generates exactly one dollar of investible funds, as it is assumed by others (Kahane, 1978; Kahane & Nye, 1975). Insurance claims are paid at the end of the period. The equity capital corresponds to the risk capital and other components that could according to regulatory statutes also be classified as risk capital are ignored.

The assets available for investment in t_0 are the aggregated premiums (π) as well as the risk capital (RC_{ε}), which are proportionally invested in N_A asset classes (j) at an expected rate of

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7 This formulation corresponds to a part of the conditional value at risk, introduced at a later point.

8 Obviously this assumption is restrictive given the ignorance of the influence of higher moments of the distribution and might be inconsistent with the commonly accepted axioms of behavior under uncertainty (Borch, 1969; Feldstein, 1969). On the other hand, studies have identified conditions under which the mean variance criterion does appear to be at least approximate valid (Merton, 1971; Samuelson, 1970).

return $E(r_j)$ with the corresponding weights w_j, where $\sum_{j=1}^{N} w_j = 1$ and r_j follow a normal distribution. The expected absolute return on the asset portfolio ($E(R)$)[9] is then determined according to (3) with $\mathbf{w'} = (w_1 \quad \cdots \quad w_N)$ and $\mathbf{R'} = (E(r_1) \quad \cdots \quad E(r_N))$.

$$E(R) = (RC_\varepsilon + \pi) \cdot \mathbf{w'R} \tag{3}$$

The income generated through the asset portfolio is paid at the end of the period in t_1. The risk of asset portfolio A is defined through the standard-deviation of returns ($\sqrt{\sigma_a^2}$), which is a composite measure of the weighted variances ($\sigma_a^2 = \mathbf{w'\Omega w}$), with the variance-covariance-matrix $\mathbf{\Omega}$ defined according to (4). Here $\rho_{i,j}$ denotes the correlation between asset i and j.

$$\mathbf{\Omega} = \begin{pmatrix} \sigma_1^2 & \rho_{2,1} \cdot \sigma_2 \cdot \sigma_1 & \cdots & \rho_{N,1} \cdot \sigma_N \cdot \sigma_1 \\ \rho_{1,2} \cdot \sigma_1 \cdot \sigma_2 & \sigma_2^2 & \cdots & \rho_{N,2} \cdot \sigma_N \cdot \sigma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,N} \cdot \sigma_1 \cdot \sigma_N & \rho_{2,N} \cdot \sigma_2 \cdot \sigma_N & \cdots & \sigma_N^2 \end{pmatrix} \tag{4}$$

The variance of the asset portfolio σ_a^2 expressed in absolute values (σ_A^2) is determined according to (5).

$$\sigma_A^2 = (RC_\varepsilon + \pi)^2 \cdot \sigma_a^2 \tag{5}$$

Insurance claims (C) are for now considered to be normally distributed ($C \sim \mathrm{N}(E(C), \sigma_C)$) and generate a cash drain at the end of the period. Given the assumed fixed mix of the insurance portfolio and the premium in t_0, the risk of the insurance portfolio is solely determined through the variance of the aggregated claims, i.e. σ_C^2. For N_C insurance lines – with a linear dependency structure amongst lines ($\rho_{Ci,Cj}$) and σ_{Ci} expressed in absolute values – σ_C is determined according to (6), with $\mathbf{\Sigma_C}$ representing the variance-covariance-matrix of the claims expressed in absolute values and $\mathbf{c'} = (\sigma_{C1} \quad \cdots \quad \sigma_{CN})$ denoting the vector of the standard deviations of the various claims.

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[9] In the following majuscules denote absolute values, minuscules denote relative values.

$$\sigma_C^2 = \mathbf{c'\Sigma_C c} = \begin{pmatrix} \sigma_{C1} & \sigma_{C2} & \cdots & \sigma_{CN} \end{pmatrix} \begin{pmatrix} 1 & \rho_{C2,C1} & \cdots & \rho_{CN,C1} \\ \rho_{C1,C2} & 1 & \cdots & \rho_{CN,C2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{C1,CN} & \rho_{C2,CN} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \sigma_{C1} \\ \sigma_{C2} \\ \vdots \\ \sigma_{CN} \end{pmatrix} \tag{6}$$

The expected company cash flow ($E(CF)$) at the end of the period generated by the two portfolios is then determined according to (7) where the relevant overall risk, i.e. the standard deviation of the cash flow (σ_{CF}), is a combination of the risk from the asset portfolio (σ_A) and the insurance risk (σ_C).

$$E(CF) = E(R) + \pi - E(C) \tag{7}$$

In absolute values σ_{CF} can be expressed through (8) with $\mathbf{\Omega_C} = \begin{pmatrix} \sigma_C^2 & \mathbf{0} \end{pmatrix}'$ and $\mathbf{COV}$ defined according to (9), where $\rho_{i,Cj}$ denotes the correlation between asset class i and insurance line j.

$$\sigma_{CF}^2 = \begin{pmatrix} (RC_\varepsilon + \pi) \cdot \mathbf{w'} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{\Omega} & \mathbf{COV} \\ \mathbf{COV} & \mathbf{\Omega_C} \end{pmatrix} \begin{pmatrix} (RC + \pi) \cdot \mathbf{w} \\ \mathbf{1} \end{pmatrix}, \tag{8}$$

with

$$\mathbf{COV} = \begin{pmatrix} \rho_{1,C1} \cdot \sigma_1 \cdot \sigma_{C1} & \cdots & \rho_{N,C1} \cdot \sigma_N \cdot \sigma_{C1} \\ \vdots & \ddots & \vdots \\ \rho_{1,CN} \cdot \sigma_1 \cdot \sigma_{CN} & \cdots & \rho_{N,CN} \cdot \sigma_N \cdot \sigma_{CN} \end{pmatrix} \tag{9}$$

Up to this point we used the RC_ε as a fixed input variable for the model, determining the investment basis of the asset-portfolio. We in the following assume the risk capital not to be fixed and focus on the determination of the required risk capital for a particular overall company risk level. To ensure solvency for $1-\varepsilon$ percent of all possible states the cash from premiums (π), the paid in risk capital (RC_ε), and the cash flow generated through the asset portfolio returns (R) are of relevance. At this point the risk capital is the residual value that has to ensure that the insurer is solvent for all instances except those excluded through the ruin probability ε.

Applying the $CVaR$ and assuming normal distributed risks allows for the formulation of (10), where we interpret the $CVaR$ as necessary RC_ε to maintain the predefined level of ruin probability with $N_{1-\varepsilon}$ denoting the $(1-\varepsilon)$-quantile of the normal distribution and φ denoting the density function of the standard normal distribution.

$$CVaR_{\varepsilon} = E(C) - \pi - E(R) + \frac{\varphi(N_{1-\varepsilon})}{\varepsilon} \cdot \sigma_{CF} = RC_{\varepsilon} \tag{10}$$

It should be noted that in equation (10) the risk capital appears on both sides of the equation, given that the capital allocation to the asset portfolio depends partially on the level of RC_{ε} as defined in (3). Solving for RC_{ε} is not straight forward. We derive (11) as the result after substituting (3) and (8) in (10) with $z = \varphi(N_{1-\varepsilon})/\varepsilon$.

$$RC_{\varepsilon} = E(C) - \pi - (RC_{\varepsilon} + \pi)\mathbf{w'R} + z\left\{\left((RC_{\varepsilon} + \pi)\cdot \mathbf{w'} \quad \mathbf{1}\right)\begin{pmatrix} \mathbf{\Omega} & \mathbf{COV} \\ \mathbf{COV} & \mathbf{\Omega_C} \end{pmatrix}\begin{pmatrix} (RC_{\varepsilon} + \pi)\mathbf{w} \\ \mathbf{1} \end{pmatrix}\right\}^{.5} \tag{11}$$

Rearranging (11) results in (12), and provided that the function has two roots we are interested in the minimum given that we intend to minimize the required RC_{ε} of the insurer. Interpreting (12) as a function of $\mathbf{w}$, i.e. $f(\mathbf{w})$, the optimal asset-liability-mix is obtained by taking the derivative of the function, i.e. $\frac{\partial f(\mathbf{w})}{\partial \mathbf{w}}$.

$$f(\mathbf{w}) = \frac{1}{2\cdot a}\cdot\left[-b \pm \left(b^2 - 4\cdot a\cdot c\right)^{.5}\right] \tag{12}$$

with:

$$a = (1 + \mathbf{w'R})^2 - z^2 \cdot \mathbf{w'\Omega w}$$

$$b = 2\cdot\left[\pi\cdot(1+\mathbf{w'R})^2 - (1+\mathbf{w'R})\cdot E(C) - z^2\cdot(\pi\cdot \mathbf{w'\Omega w} + \mathbf{w'COV})\right]$$

$$c = \pi^2\left[(1+\mathbf{w'R})^2 - z^2\cdot\mathbf{w'\Omega w}\right] - z^2\cdot\sigma_C^2 + E(C)^2 - 2\pi\left[(1+\mathbf{w'R})\cdot E(C) + z^2\cdot \mathbf{w'COV}\right]$$

Provided that weights $w_1,\ldots,w_N$ satisfy $w_1 + w_2 + \ldots w_N = 1$, the number of independent variables is reduced from N to $N-1$. Thus function $f(\mathbf{w})$ with $w_1, w_2, \ldots, w_N$ resolves to $f(\mathbf{w})$ with $w_1, w_2, \ldots, w_{N-1}, 1 - w_1 - w_2 - \ldots - w_{N-1}$ to be minimized. For $N_A = 3$ and for a fixed insurance portfolio due to existing commitments to various insurance lines, i.e. $\mathbf{COV'} = \sigma_C \cdot \left(\rho_{1,C}\cdot\sigma_1 \quad \cdots \quad \rho_{N,C}\cdot\sigma_N\right)$, an optimal asset structure can be achieved analytically. We in the following limit the number of assets to $N_A = 3$ to allow for a traceable analytical solution of (13) with $z = \varphi(N_{1-\varepsilon})/\varepsilon = 2.66521422$ corresponding to $\varepsilon = 0.01$.

$$\min_{\mathbf{w}} f(\mathbf{w}) \quad \text{s.t. } \sum_{j=1}^{N} w_j = 1 \tag{13}$$

Provided that the analytical solution is not elegant we choose to present a numerical example with a graphical depiction.[10] For the numerical example we assume three asset classes to be available for the asset allocation task with the fictive data as detailed in Table 1. We evaluate claims net of reinsurance effects with $\pi = 250$, $E(C) = 240$, and a standard deviation of the insurance portfolio of $\sigma_C = 33.6$ in absolute values.

Table 1: Assumed returns, premiums, and correlation structure of the assets and liabilities.

	Expected return/claim	Expected volatility	Correlation matrix			
	$E(r_j) / E(C)$	σ_j	A	B	C	IP
A	10.00%	20.00%	1.00			
B	6.00%	8.00%	0.35	1.00		
C	3.00%	5.50%	0.25	0.75	1.00	
Insurance portfolio (IP)	240	33.6	-0.50	-0.20	-0.10	1.00

The graphical display of (12) for various w_1 and w_2, with $-1 \le w_1 \le 1$, $-1 \le w_2 \le 1$, and $w_3 = 1 - w_1 - w_2$ is depicted in Figure 2 – numerically we derive the minimum RC_ε to be located at $w_1 = 0.3277$, $w_2 = 0.4358$, and $w_3 = 1 - w_1 - w_2 = 0.2365$.

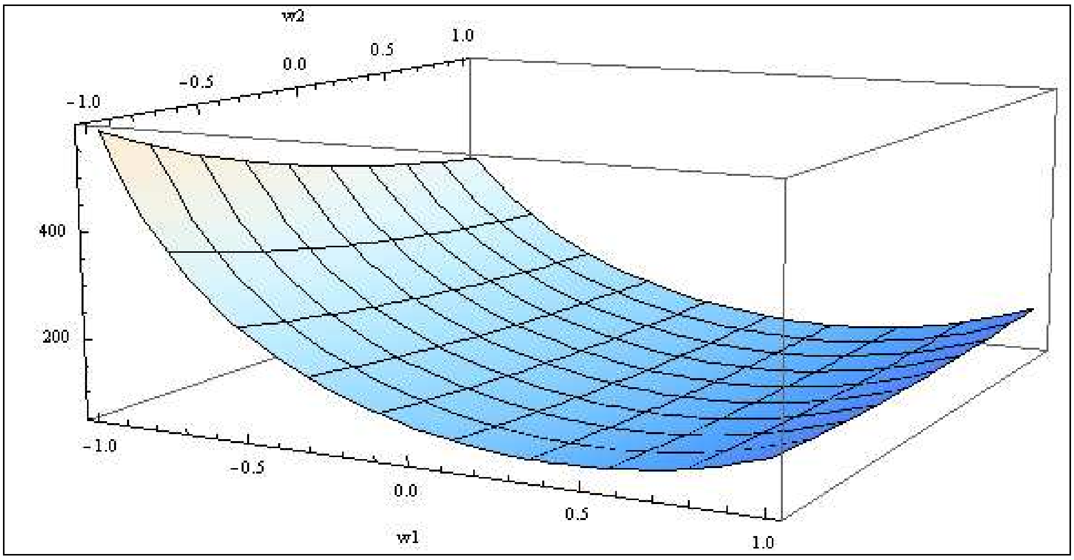

Figure 2: Graphical solution for (12) with $N_A = 3$ for $-1 \le w_1 \le 1$ and $-1 \le w_2 \le 1$, where $w_3 = 1 - w_1 - w_2$, utilizing the data as depicted in Table 1.

Depicting the relationship between RC_ε as determined through (12) and the expected cash-flow as detailed in (7), we derive Figure 3. Here the relationship of the RC_ε to grant a ruin probability

[10] For the full analytical solution contact the authors.

of $\varepsilon = 0.01$ in dependence on $E(CF)$ is illustrated, where the change of the $E(CF)$ in (7) is solely determined by the change of $E(R)$ due to changing weights w_j of the assets in the portfolio. We in addition detail the resulting RC_ε for a pure asset portfolio optimization in Figure 3, where the diversification effect originating from the insurance business is ignored.[11] It is easily seen that, as expected, the traditional asset-management does lead to a higher level of risk capital then the optimized ALM approach. In addition, there exists a minimum of RC_ε that is achieved by optimizing the asset portfolio using ALM whilst considering a particular ruin probability.

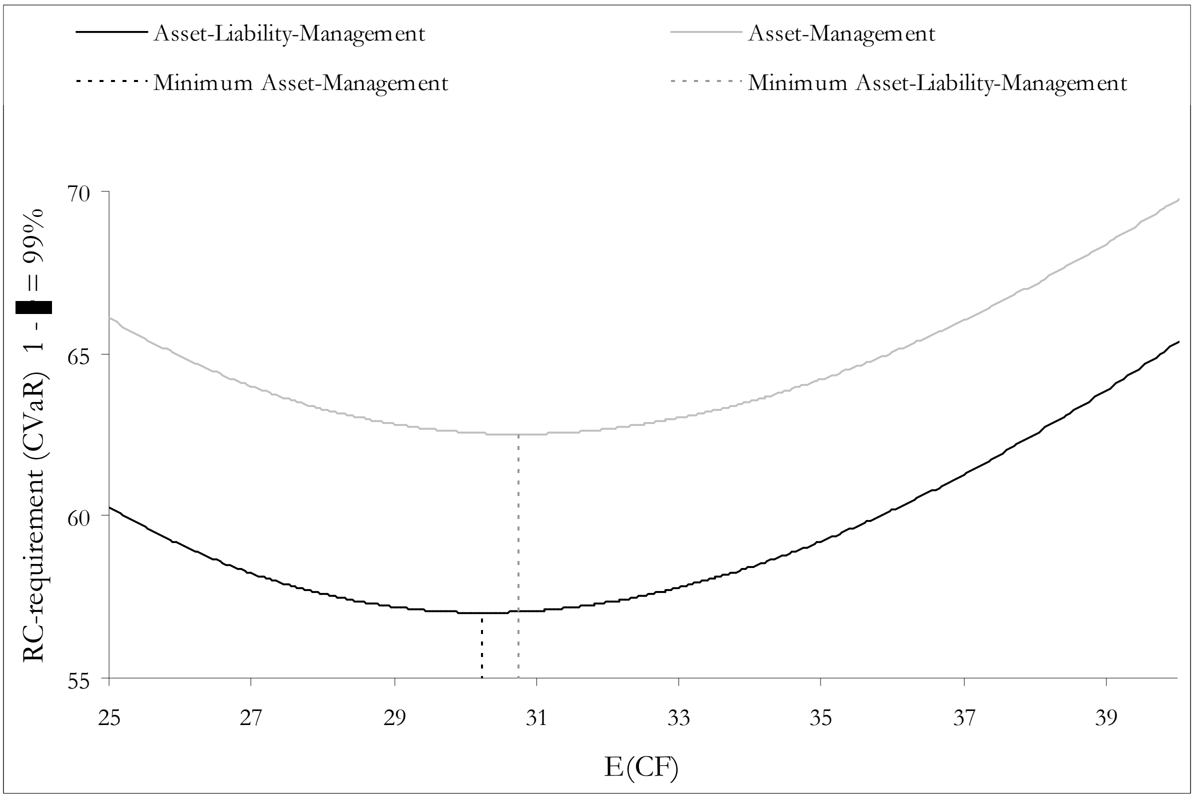

Figure 3: Risk capital, including the minimum, in dependence on the expected cash flow $E(CF)$ from the overall portfolio for asset-management and asset-liability-management derived for alternative asset combinations.

In general, we highlight that applying ALM does decrease the necessary amount of RC_ε as expected when considering diversification effects known from portfolio theory. In addition, we find that for a given ruin probability ε an optimal asset allocation exists. Overall, our results imply that when considering ALM in combination with a required level of ruin probability there exists an optimal asset-liability mix that minimizes the required level of risk capital. This finding

[11] Here the optimal weights are determined by solving (13) under the assumption that $\mathbf{COV} = 0$.

has consequences for the return on capital given that for a particular asset combination the return on capital can be maximized without inducing a change in the risk profile of the insurer, whilst fulfilling the regulatory requirements of sustaining a particular level of ruin probability. In the following we generalize the findings to more realistic distributions of claims using a simulative approach.

D. Generalization of the findings to alternative claim distributions with a non-linear dependency structure between the asset and the liability portfolio

Granted that claim distributions do not follow a normal distribution we apply the model results in a simulative approach to different distributions. We do not enter the discussion which distribution is most feasible given that depending on the problem, the distribution might be any of a large number, including: normal, lognormal, *t*, log-*t*, elliptical, hyperbolic, Pareto, Pearson-family, Johnson-family, etc. Alternatively, when dealing with extreme events, such as catastrophes, large claims, ruin probabilities for solvent institutions, extreme mortality risks, etc., an extreme value method might be best suited, such as for instance a Weibull, Gumbel, or Fréchet distribution or alternatively peaks-overthreshold theory commonly modeled utilizing a generalized Pareto distribution (Dowd & Blake, 2006). Testing the validity of the results of our model, we take a small selection of potential claim distributions into account and simulate the Normal, Weibull, Gumbel, and the Beta distribution.

In addition, we follow the literature in that solely considering linear correlations is not appropriate when modeling dependence structures between heavy-tailed and skewed insurance risks and the asset portfolio. In a simulative approach we therefore assume the claim distribution to be related to the asset portfolio through a rank correlation measure (Kendall's tau), given that rank correlations are invariant under monotonic transformations and thus not affected by the marginal distributions (McNeil, Frey, & Embrechts, 2005, pp. 206-208). We, for comparative reasons, utilize the numerical scenario, as detailed above and in Table 1, as setting and parameterize the claim distributions in such a fashion that the first two moments are identical to the scenario with normal distributed claims for all simulations. In reference to Table 1 we model an expected claim of 240 with a standard deviation of 33.60 and the associated rank correlations to the various asset classes as detailed in Table 1. We assume that assets are normal distributed and that the claim distribution follows either a Normal, Weibull, Gumbel, or Beta distribution. We implement a Weibull distribution with a shape parameter of 6, a scale parameter of 186.90, and a location parameter of -413.39, resulting in a mean of -240.00, a standard deviation of 33.60 and a skewness and kurtosis of –.37 and 3.04 respectively. The Gumbel distribution is implemented using a location parameter of $\mu = -224.88$ and a scale parameter of $s = 26.20$, resulting in a mean of -240.00, a standard deviation of 33.60 and a skewness and kurtosis of -1.14 and 5.40 respectively. The beta distribution is implemented as a four parameter distribution with two shape parameters $\alpha = 7$ and $\beta = 2.5$ and two location parameters to alter the location and scale of the distribution representing the minimum and maximum values of the distribution with $\min = -422.18$ and $\max = -174.95$, resulting in a mean of -240.00, a standard deviation of 33.60 and a skewness and kurtosis of -.6062 and 3.03 respectively.

We utilize Monte Carlo simulation to determine the optimal asset combination that ensures for a particular claim distribution and a given RC_ε that the $CVaR_\varepsilon$ does not exceed a ruin probability of $\varepsilon = 0.01$. We implement this constraint by simulating 1,000,000 runs for the various asset and the claim distributions for a particular predefined set of asset weights. We then evaluate the resulting CF distribution and the $CVaR_\varepsilon$ by slicing the resulting 1 percent tail distribution into 100 subsections and computing the mean. We apply this simulation to 15,625 permutations of asset combinations, i.e. weights, and run this set of simulations for various levels of RC_ε, i.e. we subsequently increase the level of RC_ε to be able to depict the relationship between RC_ε and $E(CF)$ as detailed in Figure 3 for alternative claim distributions. For each completed simulation run we select the set of asset weights that minimizes the difference between the predefined RC_ε and the $CVaR_\varepsilon$ for further analysis. After having conducted the simulation for a feasible range of RC_ε we obtain a scatter-plot of efficient combinations and aggregate the data as proxy for the RC_ε-function applying a non-linear 2nd degree polynomial regression analysis, the resulting functions are detailed in Table 2, including the minimum coordinates.

Granted that we base our analysis on the tail distribution of the CF distribution with $\varepsilon = 0.01$ the simulation runs need to be substantially high to obtain robust results. As a consequence there is a trade-off between accuracy and time/CPU resources. We select a middling approach that, in our opinion, provides for sufficient accuracy to illustrate the effect whilst keeping computations at a reasonable level. To illustrate the accuracy of our simulative approach we depict the analytical solution using normal distributed claims and the simulated solution for the same distribution in Table 2 and Figure 4, where the deviation of the simulated result to the analytical solution is in our opinion negligible. As illustrated in Figure 4, the evaluated relation holds for different assumptions regarding the claim distribution. In general, an optimal asset portfolio exists for any simulated distribution with varying degrees of consequences for the shareholders. As seems obvious the need for an accurate determination of the optimal asset portfolio increases with the curvature of the function, where for fairly flat structures the deviations from the minimum are almost negligible and for large curvature the accuracy seems to be of crucial importance.

Determining the optimal asset mix seems particularly important for an assumed Gumbel claim distribution, given that the risk capital is rather high and increases substantially when deviating from the optimal asset mix. These findings imply that for business units that cover events that often are approximated using a Gumbel distribution, such as catastrophes, large claims, ruin probabilities for solvent institutions, extreme mortality risks, etc. our optimization technique is of high relevance when attempting to minimize the risk capital.

Table 2: Estimated functions for the simulation of different claim distributions of the risk capital in dependence on the *E*(*CF*), including the minimum of the functions.

Assumed claim distribution for simulation runs	Estimated function of the risk capital	Minimum of function at $E(CF)$	RC_ε
Normal distribution (analytic solution)		30.25	58.34
Normal distribution (simulated solution)	$f(E(CF)) = .1042 \cdot E(CF)^2 - 6.3459 \cdot E(CF) + 155.25$	30.45	58.63
Weibull distribution	$f(E(CF)) = .1241 \cdot E(CF)^2 - 7.8246 \cdot E(CF) + 184.64$	31.52	61.30
Gumbel distribution	$f(E(CF)) = .3574 \cdot E(CF)^2 - 20.062 \cdot E(CF) + 376.98$	28.07	95.44
Beta distribution	$f(E(CF)) = .1585 \cdot E(CF)^2 - 9.9257 \cdot E(CF) + 218.80$	31.31	63.41

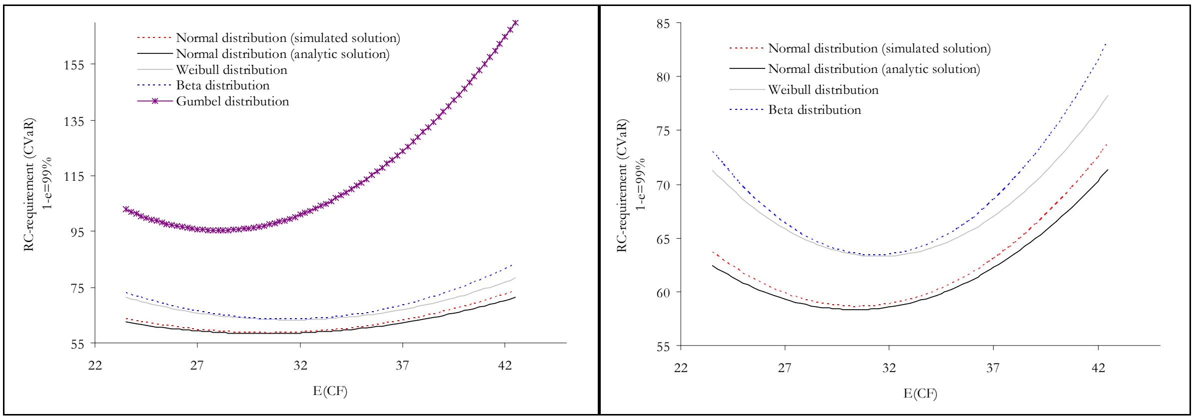

Figure 4: Illustration of the simulation results regarding the relation between $E(CF)$ and RC_ε for $\varepsilon = 0.01$ and a parameterization of the scenario according to Table 1. The left graph includes and the right graph excludes the Gumbel distribution.

As illustrated above, for any of the tested distributions a minimum of required risk capital can be determined, which maximizes the relation of cash-flow and capital, i.e. the return on capital. The more skewed the underlying claim distribution, the more capital is necessary to fulfil the ruin probability constraint. The overall result – the possibility to compute an optimal amount of required capital – holds independent of the underlying distribution assumption, but the implications regarding the severity of not reaching an optimal solution depend heavily on the claim distribution.

E. Conclusion

Motivated by the recent discussion on the topics of risk-management and risk capital requirements within Solvency II and the current financial market developments, we illustrate the close relationship between capital employed and risk-management for an insurance company, where the level of risk capital determines the company's and the associated insurance products' probability of default. We develop and evaluate an asset-liability-management model considering regulatory requirements and the specific needs of insurance companies. The model focuses on the management of assets and liabilities using modern portfolio theory and capitalizes on the resulting diversification effects from the existing dependencies of the various positions. We extend the existing literature through directly optimizing the level of required risk capital by changing the asset-liability portfolio weights whilst utilizing a tail value at risk. This is achieved through the usage of absolute values in the optimization model and the definition of the $CVaR$ as the determinant for the required risk capital. The optimization assumes a given level of ruin-probability and the traditional mean-variance optimizing investor. We in addition define the insurance portfolio to be fixed to incorporate the assumption that boldly restructuring insurance lines is not realistic in the short-term context of ALM optimization procedures. Our optimized solution provides the following results:

We show that an asset-management-only approach is inferior to an asset-liability-management approach in terms of risk-(absolute) return performance. Moreover, as a result, the level of risk capital – e.g. as required by regulators – can be reduced by capitalizing on the diversification effects from the asset as well as from the liability portfolio without incurring a deterioration of ruin probability. In addition, we find that when incorporating the $CVaR$ as constraint, ensuring a particular ruin probability, the optimization problem resolves to minimizing a quadratic function, allowing for further optimization potential. Granted that at the minimum of the function the risk capital is minimized, we find that from a managerial perspective the asset portfolio should be optimized to achieve the minimum risk capital and in consequence to increase the return on risk capital for equity holders. We confirm our results by applying the model to more realistic claim distributions (Weibull, Gumbel, and Beta) utilizing a Monte Carlo simulation. Our findings indicate that, particularly when dealing with skewed claim distributions, the risk capital changes substantially when deviating from the optimal asset mix. Overall, for any of the distributions under investigation an optimal combination of capital, risk and cash-flow can be derived, resulting in the highest possible return on capital for a given predetermined ruin probability. The provided findings are of relevance for managerial decision making, given that shareholders prefer a minimum of risk capital, i.e. tied up capital, to run operations.

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