Portfolio Selection With Spectral Risk Measures – A Really Good Choice?

Mario Brandtner*

Friedrich Schiller University of Jena, Chair of Finance, Banking, and Risk Management, Carl-Zeiss-Str. 3, D-07743 Jena

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We compare Markowitz' mean-variance portfolio selection with modern axiomatic approaches using spectral risk measures instead. Different from the previous literature, we do not only focus on the derivation of efficient frontiers, but also consider the choice of optimal portfolios within an integrating framework. The use of variance regularly yields a diversifying portfolio structure that reflects the optimal tradeoff between an investor's risk aversion and a positive risk premium. Spectral risk measures, by contrast, tend to corner solutions. If the risk free asset exists, diversification is never optimal. Instead, either the exclusive investment in the risk free asset or in the tangency portfolio obtains. Similarly, for risky assets we obtain limited diversification only. Therefore, the use of spectral risk measures for portfolio selection appears inappropriate from both a theoretical and an empirical perspective. Our findings also raise serious doubts on the use of spectral risk measures in other economic fields, such as insurance and production theory.

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*Postdoctoral research associate at the below-mentioned chair. Phone: 0049-3641-943124. E-Mail: Mario.Brandtner@wiwi.uni-jena.de.

1. Introduction

Spectral risk measures, and Conditional Value-at-Risk as the most prominent representative, have become popular risk management tools in the last decade. Originally, they have been introduced as an alternative to heavily criticized Value-at-Risk (e.g., Szegö (2002), Yamai/Yoshiba (2005)) for the assessment of solvency capital in bank regulation. In the recent literature, a change in the scope of spectral risk measures has taken place that moves the discussion away from the assessment of solvency capital ("risk"), and towards applications as (part of) an objective function in portfolio, insurance, and production theory ("decision"). In this paper, we argue that the specific properties underlying spectral risk measures are suitable for the assessment of solvency capital, but at the same time show major shortcomings that have to be taken into account if applied as an objective function. We thus suggest a serious rethinking of these approaches.

In modern bank regulation, the regulatory requirements add to a bank's objective function as a *constraint*:

$$\max_{X\in\mathcal{X}} \pi(X), \text{ s.t. } \rho(X) \leq \bar{\rho} \Leftrightarrow \tag{1}$$
$$\max_{X\in\mathcal{A}_\rho} \pi(X), \mathcal{A}_\rho = \{X \in \mathcal{X} | \rho(X) \leq \bar{\rho}\}.$$

A bank's objective function π is only allowed to be applied to those alternatives X out of its set of alternatives $\mathcal{X}$, whose solvency capital requirements $\rho(X)$ do not exceed its given solvency capital $\bar{\rho}$. In other words, the risk measure ρ restricts a bank's set of alternatives to the acceptance set $\mathcal{A}_\rho$ (Artzner et al. (1999)). In Basel II, ρ corresponds to Value-at-Risk. As a theoretically more adequate alternative, spectral risk measures like Conditional Value-at-Risk have been introduced in the literature. In particular, to overcome the paradoxical results that obtain under Value-at-Risk their definition is based on a set of axioms that reflects these regulatory issues consistently.

In the recent literature, we increasingly observe a change in the scope of spectral risk measures. As a modern portfolio selection approach, they are also applied for the derivation of (μ, ρ_ϕ)-efficient frontiers by maximization of expectation E for any given level of spectral risk $\bar{\rho}$:

$$\max_{X\in\mathcal{X}} E(X), \text{ s.t. } \rho_\phi(X) = \bar{\rho} \tag{2}$$

(e.g., Adam et al. (2008), Alexander/Baptista (2002), Alexander/Baptista (2004), Bassett et al. (2004), Bertsimas et al. (2004), De Giorgi (2002), Deng et al. (2009), Krokhmal et al. (2002)). As a rationale for this approach the authors refer abstractly to the axiomatic foundation and the popularity of spectral risk measures in financial risk management. However, they do not provide any sustainable reasoning why it is exactly the set of axioms underlying spectral risk measures that should be considered in the above optimization. In any case, the determination of (μ, ρ_ϕ)-efficient frontiers cannot be motivated as the regulatory approach given by (1). The regulator does not give any requirement on a bank's objective function, and does not require the use of a (μ, ρ_ϕ)-framework either, but only restricts the set of alternatives. For example, a bank's portfolio decisions may be based on (μ, σ^2)-preferences that are additionally restricted to a certain level of spectral risk $\bar{\rho}$ given by the regulator (e.g., ALEXANDER/BAPTISTA (2004), ALEXANDER/BAPTISTA (2006a), ALEXANDER/BAPTISTA (2006b)). Accordingly, we have to treat the determination of (μ, ρ_ϕ)-efficient frontiers as a portfolio selection approach that assumes spectral risk measures as part of a bank's, and more general, an investor's *individual* (μ, ρ_ϕ)-preferences.

On the other hand, the succeeding choice of optimal portfolios using a specific (μ, ρ_ϕ)-utility function is not subject to considerations in this literature either. However, its consideration is of great importance, since criticism on its results also questions the rationality of the preceding determination of the (μ, ρ_ϕ)-efficient frontiers. At least from the perspective of decision theory, the literature on spectral risk measures so far lacks an *integrating* portfolio selection approach that both analyzes the determination of (μ, ρ_ϕ)-efficient frontiers and the choice of optimal portfolios within *one* framework. This observation marks the starting point of our analysis.

In the recent literature on insurance and production theory, such specific (μ, ρ_ϕ)-utility functions as

$$\pi_\phi(X) = (1 - \lambda) \cdot E(X) - \lambda \cdot \rho_\phi(X), \lambda \in [0, 1] \tag{3}$$

are already widely-used as an *objective function* to model a reward-risk-tradeoff.[1] The utility function π_ϕ aggregates expectation and a (negative) spectral risk measure by a convex combination. This approach is regularly motivated by the fact that, besides the spectral risk measure ρ_ϕ, the utility function π_ϕ *itself* satisfies (except for the algebraic sign) the properties of spectral risk measures. We denote such utility functions as "spectral utility functions" below. There is a wide consensus in this literature that the axiomatic foundation *per se* is a striking advantage over non-axiomatic approaches using variance, etc. However, we argue that the specific properties underlying spectral risk measures and spectral utility

[1]In insurance theory, WAGNER (2010a) and WAGNER (2010b) apply the utility function (3) for $\rho_\phi = CVaR_\alpha$ for the determination of optimal deductible contracts. BALBÁS ET AL. (2009) apply general risk measures, which include spectral risk measures as a subclass, to the (re-)insurance problem. CAI/TAN (2007), CAI ET AL. (2008), CHEUNG (2010), DE LOURDES CENTENO/SIMÕES (2009), and TAN ET AL. (2009) study optimal (re-)insurance contracts under the stand-alone Conditional Value-at-Risk-utility function. In production theory's newsvendor model, the utility function (3) for $\rho_\phi = CVaR_\alpha$ is applied by JAMMERNEGG/KISCHKA (2007) and JAMMERNEGG/KISCHKA (2009). Moreover, an (equivalent) additive composition is used by AHMED ET AL. (2007), CHAHAR/TAAFFE (2009), and CHOI/RUSZCZYNSKI (2008). CHEN ET AL. (2009), GOTOH/TAKANO (2007), and TOMLIN/WANG (2005) use Conditional Value-at-Risk stand-alone as a utility function.

functions are inappropriate for application as an objective function in portfolio theory, and beyond in insurance and production theory as well.

In this paper, we take the perspective of decision theory rather than a practical perspective, and we show that the change in the scope of spectral risk measures from “risk” to “decision” has major shortcomings. By analyzing their application in portfolio theory, our contribution is threefold: (i) The theoretical literature on portfolio selection under spectral risk measures so far relies on normally distributed returns. We apply the so-called state preference approach instead that does not require any initial distribution. This generalization allows us to disclose restrictive portfolio structures that would otherwise remain hidden. (ii) The analyses so far have been restricted to the determination of (μ, ρ_ϕ)-efficient frontiers, and they do not cover the choice of optimal portfolios. We contribute to this open issue by applying spectral utility functions of the type (3), which will turn out to be a “natural choice” in this framework, and thus yield an integrating portfolio selection approach. (iii) Although spectral utility functions already find application as an objective function in insurance and production theory, their major shortcomings remain hidden due to the nonlinearity of insurance contracts and production schemes. On the other hand, the linearity of the portfolio selection problems allows us to disclose these shortcomings for the first time. Additionally, our contribution can be understood as a merger of recent developments in portfolio theory on one side, and recent developments in insurance and production theory on the other side.

Within this integrating framework, we find that spectral risk measures tend to corner solutions. If the risk free asset exists, diversification is never optimal. Instead, either the exclusive investment in the risk free asset or in the tangency portfolio obtains. Similarly, for risky assets we obtain limited diversification only. On the other hand, already Markowitz (1952) notes that “diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim” (p. 77). Following this view, we argue that the use of spectral risk measures appears inappropriate from both a theoretical and an empirical perspective. This has been overlooked so far, as relevant literature is focused on the derivation of (μ, ρ_ϕ)-efficient frontiers, but omits to study the succeeding choice of optimal portfolios as we do.

The paper proceeds as follows. Section 2 reviews the axiomatic foundation and the related concept of diversification underlying spectral risk measures and spectral utility functions, and introduces an initial intuition of rational portfolio structures. Section 3 derives the (μ, ρ_ϕ)-efficient frontiers and compares them with the (μ, σ^2)-efficient frontiers. Therefore, the replacement of variance by spectral risk measures is in the center of the contribution. Section 4 analyzes the choice of optimal portfolios by using spectral utility functions of the type (3). The results are confronted with those of the “hybrid model” $\pi(X) = E(X) - \frac{\lambda}{2} \cdot Var(X), \lambda \geq 0$, and with the initial intuition. We thus compare two different integrating frameworks with respect to the rationality of their induced portfolio structures. Section 5 discusses the managerial implications of our findings. Section 6 concludes.

2. Spectral risk measures versus spectral utility functions in portfolio selection

2.1. Spectral risk measures

Originally, spectral risk measures ρ_ϕ have been introduced for the assessment of solvency capital in bank regulation ("risk"). Therefore, they have to satisfy the following properties:[2]

- Monotonicity with respect to first order stochastic dominance: For $X_1, X_2 \in \mathcal{X}$ with $F_{X_1}(t) \geq F_{X_2}(t)$ and $t \in \mathbb{R}$, it holds that $\rho_\phi(X_1) \geq \rho_\phi(X_2)$.
- Translation invariance: For $X \in \mathcal{X}$ and $c \in \mathbb{R}$, it holds that $\rho_\phi(X + c) = \rho_\phi(X) - c$.
- Subadditivity: For $X_1, X_2 \in \mathcal{X}$, it holds that $\rho_\phi(X_1 + X_2) \leq \rho_\phi(X_1) + \rho_\phi(X_2)$.
- Comonotonic Additivity: For comonotonic $X_1, X_2 \in \mathcal{X}$, it holds that $\rho_\phi(X_1 + X_2) = \rho_\phi(X_1) + \rho_\phi(X_2)$.

The first two properties are straightforward requirements on monetary risk measures (Föllmer/Schied (2004), pp. 153). Monotonicity states that a financial position X_1 with a larger probability of falling below a threshold t for all $t \in \mathbb{R}$ than a financial position X_2 requires more solvency capital than X_2. Due to $\rho_\phi(X + \rho_\phi(X)) = \rho_\phi(X) - \rho_\phi(X) = 0$, translation invariance allows for the interpretation of $\rho_\phi(X)$ as necessary solvency capital.

Before we can discuss the regulatory concept of diversification underlying spectral risk measures, which will be a key issue for our argumentation, we have to introduce the notion of comonotonicity.

Definition 2.1. *Two random variables $X_1, X_2 \in \mathcal{X}$ are said to be comonotonic if*

$$(X_1(\omega_i) - X_1(\omega_j)) \cdot (X_2(\omega_i) - X_2(\omega_j)) \geq 0, \textit{for all } \omega_i, \omega_j \in \Omega. \tag{4}$$

Two random variables are comonotonic if they increase and decrease simultaneously in their state-dependent realizations. Comonotonicity thus denotes perfect dependence between two random variables. As an equivalent definition, Dhaene et al. (2002), theorem 3, proof that two random variables $X_1, X_2 \in \mathcal{X}$ are comonotonic if they can be written as non-decreasing functions $X_1 = f(Z)$ and $X_2 = g(Z)$ of the same random variable Z. Regarding this definition, comonotonicity is a generalization of perfect positive correlation. It holds that $corr(X_1, X_2) = 1$ if and only if $X_2 = a \cdot X_1 + b, a > 0, b \in \mathbb{R}$. A perfect positive correlation implies comonotonicity, but the converse is not true. For example, while comonotonicity holds between a constant and a random variable, they have a correlation coefficient of zero.

[2] The given properties differ slightly from those by Acerbi (2004) in that they do not explicitly consider law invariance and positive homogeneity, which are incorporated implicitly. Law invariance is implied by monotonicity with respect to first order stochastic dominance (Song/Yan (2009), section 5.1). Further, monotonicity and comonotonic additivity imply positive homogeneity (Schmeidler (1986), remark 1).

This, in formal terms, will be the reason why diversification between a risk free and a risky asset pays under variance, whereas it does not pay under spectral risk measures. Only under restrictive assumptions like normal distribution, comonotonicity and a perfect positive correlation are equivalent (Dhaene et al. (2002), theorem 5 and example 2).

The regulatory concept of diversification underlying spectral risk measures is captured jointly by the properties of subadditivity *and* comonotonic additivity, and it relates exclusively to the dependence structure between financial positions. Subadditivity ensures that spectral risk measures reward diversification, as a portfolio of two financial positions generally does not require more solvency capital than the sum of the solvency capital of its single positions. The diversification benefit results from an imperfect dependence structure between the financial positions X_1 and X_2 within a portfolio. In this case, a “good” realization in one state of the world of position X_1 (partially) compensates for a “bad” realization of position X_2 in the same state of the world and vice versa (co-insurance). For the special case that both financial positions are comonotonic and “good” and “bad” realizations coincide in all states of the world (as an example, consider a portfolio consisting of a security and a call option on this security), such a compensational effect does not exist. Consequently, this kind of “diversification” should not be rewarded by reduced solvency capital requirements. This is captured by the additivity of spectral risk measures for comonotonic financial positions. Comonotonic additivity in connection with monotonicity implies positive homogeneity of spectral risk measures, $\rho_\phi(\lambda \cdot X) = \lambda \cdot \rho_\phi(X), \lambda \geq 0$.

Any spectral risk measure ρ_ϕ of a random variable X is of the form

$$\rho_\phi(X) = -\int_0^1 F_X^*(p) \cdot \phi(p)\mathrm{d}p, \tag{5}$$

where $F_X^*(p) = \sup\{x \in \mathbb{R} | F_X(x) < p\}, p \in (0,1]$ are the p-quantiles of the cumulative distribution function F_X, and the risk spectrum $\phi : [0,1] \to \mathbb{R}$ satisfies the properties

- Positivity: $\phi(p) \geq 0$ for all $p \in [0,1]$,
- Normalization: $\int_0^1 \phi(p)\mathrm{d}p = 1$,
- Monotonicity: $\phi(p_1) \geq \phi(p_2)$ for all $0 \leq p_1 \leq p_2 \leq 1$

(Acerbi (2002), theorem 4.1, and more general Kusuoka (2001)). Obviously, the risk spectrum is a non-increasing density function on the unit interval. The risk spectrum assigns different weights to the p-quantiles, with smaller quantiles receiving greater weights to ensure the subadditivity of spectral risk measures. Due to

$$\rho_\phi(X) = E_{\Phi \circ F_X}(-X), \tag{6}$$

spectral risk measures are subjective probability-weighted averages of the outcomes of a random variable X. The underlying distorted cumulative distribution function results from the composition of the primitive function Φ of the risk spectrum and the cumulative distribution function F_X (Cherny (2006), theorem 3.3 and remark 3.4 (II)). For comonotonic random variables X_1 and X_2, the distorted distributions in (6) are identical, it holds that $\Phi(F_{X_1}(x_1)) = \Phi(F_{X_2}(x_1))$ for all $x_1 \in \mathbb{R}$. This linearity property for comonotonic random variables will be responsible for the restrictive portfolio structures induced by spectral risk measures.

Currently, the most widely discussed spectral risk measure is Conditional Value-at-Risk (e.g., Acerbi/Tasche (2002b), Rockafellar/Uryasev (2002)).[3] Its risk spectrum is given by

$$\phi(p) = \begin{cases} \alpha^{-1} & \text{for } 0 < p \leq \alpha \\ 0 & \text{for } \alpha < p \leq 1 \end{cases} . \tag{7}$$

The variance of a financial position X

$$Var(X) = \sigma^2 = E((X - E(X))^2) = \begin{cases} \int_{-\infty}^{\infty} (x - E(X))^2 \cdot f(x)\mathrm{d}x & \text{for } X \text{ continuous} \\ \sum_{i=1}^{n} p_i \cdot (x_i - E(X))^2 & \text{for } X \text{ discrete} \end{cases} \tag{8}$$

is not a spectral risk measure, as it satisfies none of the properties.

2.2. Spectral utility functions

According to the relevant literature, the main reason for the replacement of variance by spectral risk measures for portfolio selection is their axiomatic foundation that *per se* is seen as an advantage over non-axiomatic approaches like variance etc. For example, Acerbi/Tasche (2002a) note that "if a measure is not coherent (and spectral, the author) we just choose not to call it a risk measure at all" (p. 380).[4] To preserve these (supposed) advantages also within reward-risk-models ("decision"), the axiomatic foundation has to cover the entire reward-risk-model, and must not be restricted to the risk measure only. The literature thus regularly proposes to use the properties of spectral risk measures ("risk") also for the definition of "spectral" utility functions ("decision"). Adjusting for algebraic sign, spectral utility functions are of the form

$$\pi_\phi(X) = \int_0^1 F_X^*(p) \cdot \phi(p)\mathrm{d}p = E_{\Phi \circ F_X}(X), \tag{9}$$

where the utility spectrum ϕ is still a non-increasing density function on the unit interval. Likewise, the underlying *regulatory* concept of diversification captured jointly by superaddi-

[3]In the recent literature, the classes of exponential and power-spectral risk measures (Cotter/Dowd (2006), Dowd/Blake (2006), Dowd et al. (2008)) are being discussed. In this paper, the choice of an appropriate risk spectrum that reflects an investor's true risk preferences, although a decision problem on its own, will not be of further interest.

[4]See footnote 1 for further references.

tivity *and* comonotonic additivity remains unaffected from the change in algebraic sign when switching from spectral risk measures to spectral utility functions. The relevant literature, by contrast, regularly refers to subadditivity and superadditivity only, and omits to consider comonotonic additivity adequately.

As we can split any utility spectrum ϕ by

$$\phi(p) = \phi(1) + (1 - \phi(1)) \cdot \hat{\phi}(p), \text{ where } \hat{\phi}(p) = \frac{\phi(p) - \phi(1)}{1 - \phi(1)}, \phi(1) \in [0,1], \tag{10}$$

the corresponding spectral utility function becomes a reward-risk-model of the form

$$\pi_\phi(X) = \phi(1) \cdot E(X) + (1 - \phi(1)) \cdot \pi_{\hat{\phi}}(X) = \phi(1) \cdot E(X) - (1 - \phi(1)) \cdot \rho_{\hat{\phi}}(X). \tag{11}$$

The aggregation of expectation and a (negative) spectral risk measure by a convex combination thus appears as a "natural choice": If an investor accepts the axioms of spectral utility functions for portfolio selection, she has to (i) search for (μ, ρ_ϕ)-efficient portfolios, and (ii) choose her optimal portfolio by applying the (linear) spectral (μ, ρ_ϕ)-utility function (11). In this sense, the replacement of variance by spectral risk measures for the derivation of (μ, ρ)-efficient frontiers[5] and the choice of optimal portfolios using a spectral utility function form an integrating portfolio selection approach; they are two sides of the same coin. Or, conversely: As recent approaches on portfolio selection justify their search for (μ, ρ_ϕ)-efficient frontiers with the (supposed) advantages of the axiomatic foundation of spectral risk measures, they implicitly accept their underlying axioms. Accordingly, the choice of optimal portfolios then has to be based on spectral utility functions. This view is in line with Acerbi/Simonetti (2002), who note that "minimizing a Spectral Measure is already in some sense "minimizing risks and maximizing returns at the same time"" (p. 10). However, they only focus on optimization procedures, and they do not consider portfolio selection problems within an integrating framework as we do.

A prominent example is the convex combination of expectation and (negative) Conditional Value-at-Risk

$$\pi_{\alpha,\lambda}(X) = (1 - \lambda) \cdot E(X) - \lambda \cdot CVaR_\alpha(X), \alpha \in (0,1], \lambda \in [0,1] \tag{12}$$

(e.g., Acerbi/Simonetti (2002), example 4.4), whose utility spectrum is given by

$$\phi_{\alpha,\lambda}(p) = \begin{cases} (1 - \lambda) + \lambda \cdot \alpha^{-1} & \text{for } 0 < p \leq \alpha \\ (1 - \lambda) & \text{for } \alpha < p \leq 1 \end{cases} . \tag{13}$$

Basically, the concept of spectral risk measures and spectral utility functions is not new. The idea of distorted probabilities has been introduced in the literature by the dual theory

[5]Below, we use ρ as a placeholder for variance σ^2 and spectral risk measures ρ_ϕ. The term "(μ, ρ)-efficient frontiers", for example, stands for (μ, σ^2)- and (μ, ρ_ϕ)-efficient frontiers.

of choice, which provides a representation of the form

$$\pi_D(X) = \int_0^1 F_X^*(p)\mathrm{d}v(p) = E_{v\circ F_X}(X) \tag{14}$$

(e.g., Denneberg (1988), Roell (1987), Yaari (1987)).[6] The representation π_D is characterized by the dual utility function v that distorts the physical cumulative distribution function F_X. Obviously, the dual utility function coincides with the primitive function Φ of the utility spectrum. For the representation of risk aversion, the dual utility function has to be concave with $v(0) = 0, v(1) = 1$, recovering the class of spectral risk measures and spectral utility functions, respectively.

2.3. Portfolio selection problems and their initial intuition

We will analyze the consequences of the change in the scope of spectral risk measures and spectral utility functions from "risk" to "decision" by applying them to simple portfolio selection problems: An investor can split her initial wealth W_0 between different assets. The return from this investment (i.e. the final wealth) is given by a random variable $X \in \mathcal{X}$ that stems from the below settings.

- *Setting 1*: The optimal allocation between two risky assets X_1 and X_2, i.e. $\mathcal{X} = \{\gamma \cdot X_1 + (1-\gamma)\cdot X_2 | \gamma \in [0,1]\}$.[7] We assume the risky assets to be strictly (μ,ρ)-efficient, i.e. $E(X_1) < E(X_2) \wedge \rho(X_1) < \rho(X_2)$.

- *Setting 2*: The optimal allocation between two risky assets X_1 and X_2, and a risk free asset X_0, i.e. $\mathcal{X} = \{\beta \cdot (\gamma \cdot X_1 + (1-\gamma)\cdot X_2) + (1-\beta)\cdot X_0 | \beta, \gamma \in [0,1]\}$. Again, we assume the risky assets to be strictly (μ,ρ)-efficient. Moreover, we restrict the correlation coefficient to $corr(X_1, X_2) \in (-1, 1)$ to ensure that one cannot construct an additional risk free asset from the risky assets. Further assumptions on the return of the risk free asset will be made in the respective sections.

We restrict our analysis on two risky assets, as the effects of comonotonicity on the portfolio structure are in the core of the contribution. Comonotonicity in turn is a property that is shared by two assets. For more than two risky assets, these effects are regularly overlaid by dependencies with the other assets. On the other hand, the simplicity of the above settings allows us to disclose restrictive portfolio structures that so far have been overlooked in more general settings. Moreover, our main goal is to study the choice of optimal portfolios within an integrating (μ, ρ_ϕ)-framework from the perspective of decision theory for the first time,

[6] Wang et al. (1997) derive an identical representation of so-called distortion measures for the pricing of insurance contracts (see also Wang (2000), Young (1999)).

[7] $X_1 := W_0 \cdot (1 + R_1)$ and $X_2 := W_0 \cdot (1 + R_2)$ denote the returns from investing the initial wealth in asset 1 and 2. The proportions $\gamma := \frac{W_{0,1}}{W_0}$ and $1-\gamma := \frac{W_0 - W_{0,1}}{W_0}$ denote the fractions of the initial wealth that are invested in asset 1 and 2.

rather than to generalize already existing results on (μ, ρ_ϕ)-efficient frontiers. Anyhow, our main results will also hold for m risky assets.

We will confront the portfolio structures under spectral risk measures with those of variance as risk measure (Markowitz (1952)). For the evaluation of the rationality of using spectral risk measures and spectral utility functions for portfolio selection, we give the following precise initial intuition under which conditions we consider a resulting portfolio structure to be rational:

- Diversification first of all is the result of the optimal tradeoff between an investor's risk aversion and a positive risk premium[8]. For example, De Giorgi (2005) notes that classical portfolio theory "reduces the portfolio choice to a set of two criteria, reward and risk, with possible tradeoff analysis" (p. 895). This tradeoff is reflected by the (μ, ρ)-utility functions that consist of a reward measure μ and a risk measure ρ. The dependence structure between the assets only has an indirect impact on this tradeoff by affecting the risk measure, whereas the reward measure in form of expectation (De Giorgi (2005), theorem 4.1) remains unaffected from a change in the dependencies. Note that this is contrary to spectral risk measures and spectral utility functions that consider the dependence structure as the only source of positive diversification benefits.

- If, as stated above, the tradeoff between reward and risk marks the origin of diversification, and the corresponding utility function is "rich" enough to cover all degrees of risk aversion between risk neutrality and infinite risk aversion,[9] the following should hold: (i) The optimal portfolio structure should regularly yield interior solutions $\beta^*, \gamma^* \in (0, 1)$ for any dependence structure. This should also hold for comonotonic assets, as an investor may prefer a ((μ, ρ)-efficient) risk-return profile that lies in the interior of the comonotonic assets' risk-return profiles. (ii) As a sensitivity requirement, a marginal change in the risk aversion and in the risk premium should yield a marginal change in the optimal portfolio structure as well. For the limiting cases of infinite risk aversion and sufficiently large risk premia, the exclusive investment in the most riskless and in the riskiest asset, respectively, should be optimal. (iii) Finally, as an optimal alternative, any (μ, ρ)-efficient alternative should come into consideration.

If a portfolio structure satisfies this intuition, we refer to it as *full diversification*. Otherwise, we speak of *limited diversification*.

[8]In the case of two risky assets the risk premium denotes the difference between the expected returns.

[9]This property is satisfied by the hybrid model and spectral utility functions, whose induced indifference curves cover any slope between zero and infinity (see section 4.1).

3. (μ, σ^2)-efficient versus (μ, ρ_ϕ)-efficient portfolios

3.1. State preference approach

3.1.1. Framework and comonotonic subsets of alternatives

The determination of a portfolio's risk hinges crucially on the dependence structure between the risky assets. Whereas a portfolio's variance can be calculated directly from its (basic) assets' variances and the correlation coefficient, spectral risk measures due to their rank-dependency require the complete dependence structure to determine a portfolio's spectral risk. The theoretical literature so far thus mostly relies on normally distributed returns, as the correlation coefficient in this case captures the dependence structure completely. We refrain from this assumption, and we introduce the so-called "state preference approach" instead that characterizes the assets $X : \Omega \to \mathbb{R}$ via their state-dependent realizations $X = (X(\omega_1), \ldots, X(\omega_n))' = (x_1, \ldots, x_n)'$ and the corresponding vector of the probabilities of the states of the world $P = (P(\omega_1), \ldots, P(\omega_n))' = (p_1, \ldots, p_n)'$, i.e. any alternative is given by the pair (X, P). This approach captures the dependence structure completely by the vectors X, and variance and spectral risk measures can be calculated directly from X. The argumentation first remains restricted to a finite state space, as certain portfolio structures "get lost" in the case of infinitely many states (e.g., Alexander et al. (2007)).

To our best knowledge, we are the first to apply the state preference approach to portfolio selection problems under spectral risk measures. Different from the previous theoretical literature that relies on normally distributed returns, we do not require any initial distribution. Our approach thus is more general and, at least from the perspective of decision theory, superior to this literature in that it allows us to disclose restrictive portfolio structures that would otherwise remain partially hidden. A relaxation of the assumption of normally distributed returns has also been suggested by Adam et al. (2008), who note that "a general theory involving any arbitrary portfolio distributions and risk measures seems out of scope. We think that it is more insightful to consider a realistic case study, where portfolio returns are actually non Gaussian" (p. 1871). Different from their empirical study, we provide a theoretical approach that gives us a more detailed picture of the portfolio structures than normally distributed returns would do.

As spectral risk measures and spectral utility functions are comonotonic additive, comonotonic subsets of alternatives become an essential part of the analysis. The state preference approach allows us to make the comonotonic subsets of alternatives explicit via their state-dependent realizations. Let

$$\mathcal{X} = \left\{ X_\gamma = \gamma \cdot X_1 + (1-\gamma) \cdot X_2 = \begin{pmatrix} \gamma \cdot x_{1_1} + (1-\gamma) \cdot x_{2_1} \\ \vdots \\ \gamma \cdot x_{1_n} + (1-\gamma) \cdot x_{2_n} \end{pmatrix} \middle| \gamma \in [0,1] \right\} \tag{15}$$

be the set of alternatives based on the two risky assets. The boundaries of the comonotonic subsets of alternatives are given by

$$\gamma_{ij} := \frac{x_{2_i} - x_{2_j}}{(x_{2_i} - x_{2_j}) - (x_{1_i} - x_{1_j})}, i = 1, \ldots, n-1, j = 2, \ldots, n, i < j. \tag{16}$$

We obtain the proportions (16) by equalizing any two portfolio realizations and solving for γ. Therefore, any γ_{ij} denotes a portfolio where the ordering of the realizations changes. Rearranging the proportions with respect to size yields the following k comonotonic subsets of alternatives ($\gamma_{ij} \notin [0,1]$ are excluded by short-sale constraint):

$$\{X_\gamma | \gamma \in [0, \gamma_{ij,1:k}]\}, \{X_\gamma | \gamma \in (\gamma_{ij,1:k}, \gamma_{ij,2:k}]\}, \ldots, \{X_\gamma | \gamma \in (\gamma_{ij,k:k}, 1]\}. \tag{17}$$

The number of comonotonic subsets depends mainly on the number of states of the world. For $n \to \infty$, k may (but need not necessarily) tend to infinity.

Moreover, for one risk free and one risky asset, the complete set of alternatives

$$\mathcal{X} = \{X_\beta = \beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0 | \beta \in [0,1]\} \tag{18}$$

is comonotonic.

Next, we give the definitions of the (μ, ρ)-boundary and the (μ, ρ)-efficient frontier.

Definition 3.1. *A portfolio $X \in \mathcal{X}$ belongs to the (μ, ρ)-boundary if for some expected return $\bar{E} \in \mathbb{R}$ it has minimum risk ρ.*

Definition 3.2. *A portfolio $X \in \mathcal{X}$ belongs to the (μ, ρ)-efficient frontier if no portfolio $\bar{X} \in \mathcal{X}$ exists with $E(\bar{X}) \geq E(X)$ and $\rho(\bar{X}) \leq \rho(X)$, where at least one of the inequalities is strict.*

As it is common in portfolio selection, we illustrate their derivation in the (ρ, μ)-planes.[10] Different from the previous literature, we are not only interested in the (μ, ρ)-efficient frontiers itself, but especially in their *shape* (e.g., (piecewise) linear, (strictly) concave).

3.1.2. Two risky assets

We now turn to the derivation of the (μ, ρ)-boundaries and the (μ, ρ)-efficient frontiers. As we restrict our analysis on two risky assets, the complete set of alternatives $X_\gamma = \gamma \cdot X_1 + (1-\gamma) \cdot X_2, \gamma \in [0,1]$ belongs to the $(\mu, \rho)_1$-boundaries[11].

[10] For variance, we give the illustration in the (σ^2, μ)-plane instead of the commonly used (σ, μ)-plane. This is due to the fact that the choice of optimal portfolios in section 4 requires the respective risk measures on the abscissa.

[11] The subscript 1 (Setting 1) indicates that the (μ, ρ)-boundaries and the (μ, ρ)-efficient frontiers are entirely composed of risky assets. In the case of an additional risk free asset, we use the subscript 2 (Setting 2).

Let us start with (μ, σ^2)-preferences. We obtain the $(\mu, \sigma^2)_1$-boundary by solving the portfolio's expected return for the proportion γ and plugging it into its variance:

$$\begin{aligned} Var(X_\gamma) = &\left(\frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)}\right)^2 \cdot a + 2 \cdot \frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)} \cdot b + c, \\ a &= Var(X_1) + Var(X_2) - 2 \cdot \sqrt{Var(X_1)} \cdot \sqrt{Var(X_2)} \cdot corr(X_1, X_2), \\ b &= \sqrt{Var(X_1)} \cdot \sqrt{Var(X_2)} \cdot corr(X_1, X_2) - Var(X_2), \\ c &= Var(X_2). \end{aligned} \tag{19}$$

The $(\mu, \sigma^2)_1$-boundary is located on a parabola that opens to the right (see figure 1). The $(\mu, \sigma^2)_1$-efficient frontier lies on the upper branch of the parabola starting from the minimum-variance-portfolio[12].

Proposition 3.3. *Let $\mathcal{X}$ as in Setting 1. Then the following holds:*

1. *The minimum-variance-portfolio is given by $\gamma_{MVP} = \min\{\gamma_{GMVP}; 1\}, \gamma_{GMVP} = -\frac{b}{a}$.*
2. *The $(\mu, \sigma^2)_1$-efficient frontier contains all portfolios $\gamma \in [0, \gamma_{MVP}]$ and lies on a strictly concave curve for any correlation coefficient, $corr(X_1, X_2) \in [-1, 1]$.*
3. *For comonotonic X_1 and X_2, the $(\mu, \sigma^2)_1$-efficient frontier lies on a strictly concave curve.*

See the appendix for the proof. From the mutual fund theorem we know that the $(\mu, \sigma^2)_1$-boundary for m risky assets can be generated by any two distinct $(\mu, \sigma^2)_1$-boundary-portfolios (Merton (1972), section 3). Therefore, the above-stated strict concavity of the $(\mu, \sigma^2)_1$-efficient frontier preserves for m risky assets. The formal condition for a portfolio belonging to the $(\mu, \sigma^2)_1$-boundary in this case is given by Merton (1972), pp. 1853.

Let us now consider (μ, ρ_ϕ)-preferences. We obtain the $(\mu, \rho_\phi)_1$-boundary by writing the portfolio's expected return as a function of its spectral risk.

In a first step, we analyze a comonotonic subset of alternatives $X_\gamma, \gamma \in [\gamma_d, \gamma_u]$ as given in (17). Let $\delta := \frac{\gamma - \gamma_d}{\gamma_u - \gamma_d} \in [0, 1]$, then due to comonotonic additivity and positive homogeneity we get

$$\begin{aligned} \rho_\phi(X_\gamma) = \rho_\phi(\delta \cdot X_{\gamma_d} + (1 - \delta) \cdot X_{\gamma_u}) &= \delta \cdot \rho_\phi(X_{\gamma_d}) + (1 - \delta) \cdot \rho_\phi(X_{\gamma_u}) \Leftrightarrow \\ \delta &= \frac{\rho_\phi(X_\gamma) - \rho_\phi(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})}. \end{aligned} \tag{20}$$

[12] The subscripts MVP and MSP, respectively, denote the minimum-variance-portfolio and the minimum-spectral risk-portfolio in the presence of short-sale-constraints. The subscripts $GMVP$ and $GMSP$ (global) denote the respective minimum-risk-portfolios in the absence of short-sale-constraints.

Substituting for the proportion δ, the portfolio's expected return becomes

$$\begin{aligned} E(X_\gamma) &= \delta \cdot E(X_{\gamma_d}) + (1-\delta) \cdot E(X_{\gamma_u}) \\ &= \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \cdot \rho_\phi(X_\gamma) - \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \cdot \rho_\phi(X_{\gamma_u}) + E(X_{\gamma_u}). \end{aligned} \tag{21}$$

For comonotonic subsets of alternatives, the portfolio's expected return is a linear function of its spectral risk. If the comonotonic subset of alternatives is (μ, ρ_ϕ)-efficient, i.e. $E(X_{\gamma_d}) \geq E(X_{\gamma_u}) \wedge \rho_\phi(X_{\gamma_d}) \geq \rho_\phi(X_{\gamma_u})$, (21) is linearly increasing, and linearly decreasing otherwise.

Regarding the complete set of alternatives $X_\gamma, \gamma \in [0,1]$ due to subadditivity and positive homogeneity the portfolio's spectral risk is convex on $\mathcal{X}$:

$$\rho_\phi(\gamma \cdot X_{\gamma_1} + (1-\gamma) \cdot X_{\gamma_2}) \leq \gamma \cdot \rho_\phi(X_{\gamma_1}) + (1-\gamma) \cdot \rho_\phi(X_{\gamma_2}), \tag{22}$$

for all $X_{\gamma_1}, X_{\gamma_2} \in \mathcal{X}, \gamma \in [0,1]$. As the portfolio's expected return and the proportion γ are linearly connected, the portfolio's spectral risk is also a convex function of its expected return that according to (21) is piecewise linear for comonotonic subsets of alternatives. The $(\mu, \rho_\phi)_1$-boundary thus is located on a piecewise linear and overall convex curve that opens to the right (see figure 1). The $(\mu, \rho_\phi)_1$-efficient frontier lies on the upper branch of the $(\mu, \rho_\phi)_1$-boundary starting from the minimum-spectral risk-portfolio. We resume the results from the above argumentation in the following proposition.

Proposition 3.4. *Let $\mathcal{X}$ as in Setting 1. Then the following holds:*

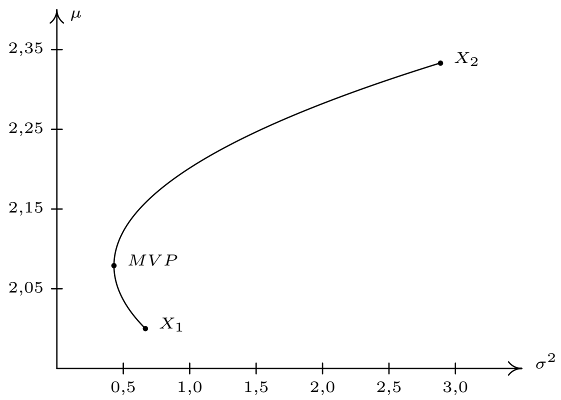
1. *The minimum-spectral risk-portfolio lies in the set $\gamma_{MSP} \in \{\gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}, 1\}$.*
2. *The $(\mu, \rho_\phi)_1$-efficient frontier contains all portfolios $\gamma \in [0, \gamma_{MSP}]$ and lies on a concave curve that is piecewise linear for comonotonic subsets of alternatives as given in (17).*
3. *For comonotonic X_1 and X_2, the $(\mu, \rho_\phi)_1$-efficient frontier contains all portfolios $\gamma \in [0,1]$ and lies on a straight line.*

As comonotonicity only implies $corr(X_1, X_2) \in [0,1]$, the $(\mu, \rho_\phi)_1$-efficient frontier may also be a straight line for any $corr(X_1, X_2) \in [0,1]$.

More general, the piecewise linear and overall convex shape of the $(\mu, \rho_\phi)_1$-boundary preserves for m risky assets.[13]

We give the following example for numerical demonstration.

[13] To proof this, we consider two comonotonic subsets of alternatives $\gamma \cdot X_{\gamma_1} + (1-\gamma) \cdot X_{\gamma_3}, \gamma \in [0,1]$ and $\gamma \cdot X_{\gamma_2} + (1-\gamma) \cdot X_{\gamma_3}, \gamma \in [0,1]$ with $E(X_{\gamma_1}) < E(X_{\gamma_3}) < E(X_{\gamma_2})$ and $\rho_\phi(X_{\gamma_1}) < \rho_\phi(X_{\gamma_2}) < \rho_\phi(X_{\gamma_3})$ that together constitute a concave curve that is piecewise linear and opens to the left. The subset of alternatives $\gamma \cdot X_{\gamma_1} + (1-\gamma) \cdot X_{\gamma_2}, \gamma \in [0,1]$, by contrast, constitutes a convex curve that opens to the right. This new subset of alternatives dominates the initially given subsets of alternatives and restores convexity and piecewise linearity of the $(\mu, \rho_\phi)_1$-boundary.



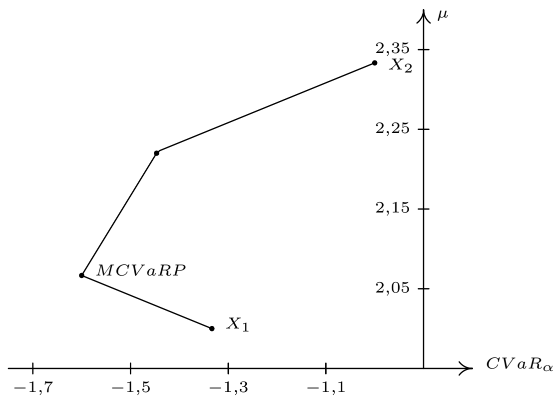


Figure 1: $(\mu, \sigma^2)_1$- versus $(\mu, CVaR_\alpha)_1$-boundary with two risky assets

Example 3.5. An investor can split her initial wealth between two risky assets with the state-dependent returns

$$X_1 = \begin{cases} 1 & P(\omega_1) = 1/3 \\ 2 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases} \text{ and } X_2 = \begin{cases} 4 & P(\omega_1) = 1/3 \\ 0 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases}.$$

As risk measures, she applies variance and Conditional Value-at-Risk at the confidence level $\alpha = 0{,}5$. The risky assets are (μ, σ^2)-efficient, as $E(X_1) = 2 < 2{,}34 = E(X_2), Var(X_1) = 0{,}67 < 2{,}89 = Var(X_2)$, and they are $(\mu, CVaR_\alpha)$-efficient, as $CVaR_\alpha(X_1) = -1{,}34 < -1 = CVaR_\alpha(X_2)$ holds.

Figure 1 shows the $(\mu, \sigma^2)_1$-boundary. The minimum-variance-portfolio is given by $X_{0{,}7631} = (1{,}53; 1{,}71; 3)'$, and the $(\mu, \sigma^2)_1$-efficient frontier contains all portfolios $X_\gamma, \gamma \in [0; 0{,}7631]$.

Further, figure 1 shows the $(\mu, CVaR_\alpha)_1$-boundary. The $(\mu, CVaR_\alpha)_1$-efficient frontier contains all portfolios $X_\gamma, \gamma \in [0; 0{,}8]$, with $X_{0{,}8} = (1{,}6; 1{,}6; 3)$ being the minimum-Conditional Value-at-Risk-portfolio.[14] The linear segments correspond to the portfolios $\gamma \in [0; 0{,}3333]$ $(x_2 \le x_3 \le x_1)$, $\gamma \in (0{,}3333; 0{,}8]$ $(x_2 \le x_1 \le x_3)$, and $\gamma \in (0{,}8; 1]$ $(x_1 \le x_2 \le x_3)$. The corners $X_{0{,}3333} = (3; 0{,}67; 3)'$ and $X_{0{,}8} = (1{,}6; 1{,}6; 3)'$ are characterized by having at least two identical state-dependent realizations.

3.1.3. One risk free and two risky assets

In a first step, we will give the derivation of the $(\mu, \rho)_2$-efficient frontiers for one risk free asset X_0 and only one risky asset $X_{\bar\gamma}$, i.e. the set of alternatives reads $X_{\beta,\bar\gamma} = \beta \cdot X_{\bar\gamma} + (1-\beta) \cdot X_0, \beta \in [0, 1]$. Afterwards, we will treat the risky asset $X_{\bar\gamma}$ as a $(\mu, \rho)_1$-efficient portfolio composed of the two risky (basic) assets.

Before we start with the analysis, we have to make additional assumptions on the risk

[14]In this case, the minimum-CVaR-portfolio is not $(\mu, \sigma^2)_1$-efficient. This demonstrates the incompatibility of variance and Conditional Value-at-Risk, which can lead to “perverse situations” in a bank's risk management (Alexander/Baptista (2004), Alexander/Baptista (2006a)).

free asset. In the case of (μ, σ^2)-preferences, we stay in line with the literature and assume that $X_0 < E(X_{GMVP})$ to ensure that the risk free asset lies below the intersection of the asymptote of the $(\mu, \sigma^2)_1$-efficient frontier with the ordinate. To attain the same property for (μ, ρ_ϕ)-preferences, we first have to assume that $X_0 < E(X_A)$, with $E(X_A)$ being the expected return at the intersection of the asymptote of the $(\mu, \rho_\phi)_1$-efficient frontier with the bisector of the second quadrant.[15] Note that risk free assets lie on this bisector, as a change in the expected return of the risk free asset due to translation invariance simultaneously affects its spectral risk to the same extent. Therefore, as an additional condition $\rho_\phi(X_0) < \rho_\phi(X_{GMSP})$ must hold, so that altogether we get $-\rho_\phi(X_{GMSP}) < X_0 < E(X_A) = -\rho_\phi(X_A)$.

Again, the derivation of the $(\mu, \sigma^2)_2$-boundary with respect to the set of alternatives $X_{\beta,\bar{\gamma}}, \beta \in [0, 1]$ requires to solve the portfolio's variance for the proportion β and to plug it into its expected return, which yields

$$E(X_{\beta,\bar{\gamma}}) = \frac{E(X_{\bar{\gamma}}) - X_0}{\sqrt{Var(X_{\bar{\gamma}})}} \cdot \sqrt{Var(X_{\beta,\bar{\gamma}})} + X_0. \tag{23}$$

The portfolio's expected return is a strictly concave (square-root) function of its variance.

Generally, any $(\mu, \sigma^2)_1$-efficient portfolio can serve as risky asset $X_{\bar{\gamma}}$ in the above considerations. The only combination (23) that is $(\mu, \sigma^2)_2$-efficient consists of the risk free asset X_0 and the $(\mu, \sigma^2)_1$-efficient portfolio X_{T,σ^2} that touches the parabola (19) in only a single point, and thus is called tangency portfolio (see figure 2).

Proposition 3.6. *Let $\mathcal{X}$ as in Setting 2 and $X_0 < E(X_{GMVP})$. Then the following holds:*

1. *The $(\mu, \sigma^2)_2$-efficient frontier lies on a strictly concave curve between the risk free asset and the tangency portfolio.*
2. *The tangency portfolio is given by*

$$\gamma_{T,\sigma^2} = \max\left\{\min\left\{\frac{(E(X_2) - X_0) \cdot b - (E(X_1) - E(X_2)) \cdot c}{(E(X_1) - E(X_2)) \cdot b - (E(X_2) - X_0) \cdot a}; 1\right\}; 0\right\}. \tag{24}$$

See the appendix for the proof. The proposition gives us TOBIN's separation theorem (TOBIN (1958)): Any $(\mu, \sigma^2)_2$-efficient portfolio is a linear combination of the risk free asset and the tangency portfolio. An investor's individual risk aversion only affects the proportions of the initial wealth that are invested in these assets. Note that for any given parameters $E(X_1), E(X_2), Var(X_1), Var(X_2)$, and $corr(X_1, X_2) \in (-1, 1)$, there exists a

[15] In formal terms, the intersection is given by

$$E(X_A) = -\rho_\phi(X_A) = \frac{E(X_{\gamma_{\min}}) - z \cdot \rho_\phi(X_{\gamma_{\min}})}{z + 1},$$

where $\gamma_{\min} = \min\{\gamma_{ij}, i = 1, \ldots, n-1, j = 2, \ldots, n, i < j\}$, and $z = \frac{E(X_{\gamma_{\min}+\Delta}) - E(X_{\gamma_{\min}})}{\rho_\phi(X_{\gamma_{\min}+\Delta}) - \rho_\phi(X_{\gamma_{\min}})}, \Delta < 0$.

corresponding value X_0, so that any $(\mu, \sigma^2)_1$-efficient portfolio can serve as tangency portfolio, i.e. $\gamma_{T,\sigma^2} \in [0,1]$.

TOBIN's separation theorem also holds for m risky assets. In this case, the composition of the tangency portfolio is given by MERTON (1972), theorem 2. Especially, the $(\mu, \sigma^2)_2$-efficient frontier according to (23) is still a strictly concave curve between the risk free asset and the tangency portfolio.

A similar argumentation applies for the $(\mu, \rho_\phi)_2$-boundary. As the set of alternatives $X_{\beta,\bar{\gamma}}, \beta \in [0,1]$ is comonotonic, and spectral risk measures are comonotonic additive and positive homogeneous, we can solve the portfolio's spectral risk for the proportion β as

$$\begin{aligned} \rho_\phi(X_{\beta,\bar{\gamma}}) = \rho_\phi(\beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0) = \beta \cdot \rho_\phi(X_{\bar{\gamma}}) + (1-\beta) \cdot \rho_\phi(X_0) \Leftrightarrow \\ \beta = \frac{\rho_\phi(X_{\beta,\bar{\gamma}}) - \rho_\phi(X_0)}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)} \end{aligned} \tag{25}$$

and substitute it into its expected return:

$$\begin{aligned} E(X_{\beta,\bar{\gamma}}) &= \beta \cdot E(X_{\bar{\gamma}}) + (1-\beta) \cdot X_0 \\ &= \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)} \cdot \rho_\phi(X_{\beta,\bar{\gamma}}) - \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)} \cdot \rho_\phi(X_0) + X_0. \end{aligned} \tag{26}$$

The portfolio's expected return is linearly increasing in its spectral risk.

Again, any $(\mu, \rho_\phi)_1$-efficient portfolio $X_{\bar{\gamma}}$ can serve as risky asset. However, the only combination that is $(\mu, \rho_\phi)_2$-efficient consists of the risk free asset X_0 and the $(\mu, \rho_\phi)_1$-efficient portfolio where (26) is a tangent to the $(\mu, \rho_\phi)_1$-efficient frontier, X_{T,ρ_ϕ} (tangency portfolio) (see figure 2). We resume the results from the above argumentation in the following proposition.

Proposition 3.7. *Let $\mathcal{X}$ as in Setting 2 and $-\rho_\phi(X_{GMSP}) < X_0 < E(X_A)$. Then the following holds:*

1. *The $(\mu, \rho_\phi)_2$-efficient frontier lies on a straight line between the risk free asset and the tangency portfolio.*

2. *The tangency portfolio lies in the set $\gamma_{T,\rho_\phi} \in \{0, \gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}, 1\}$.*

Apart from that TOBIN's separation theorem remains valid. As the shape of the $(\mu, \rho_\phi)_1$-efficient frontier preserves for m risky assets, TOBIN's separation theorem with respect to the boundaries of comonotonic subsets of alternatives also holds for one risk free and m risky assets. Especially, the $(\mu, \rho_\phi)_2$-efficient frontier according to (26) is still a straight line between the risk free asset and the tangency portfolio.

In the following example, we add the risk free asset to the example 3.5.

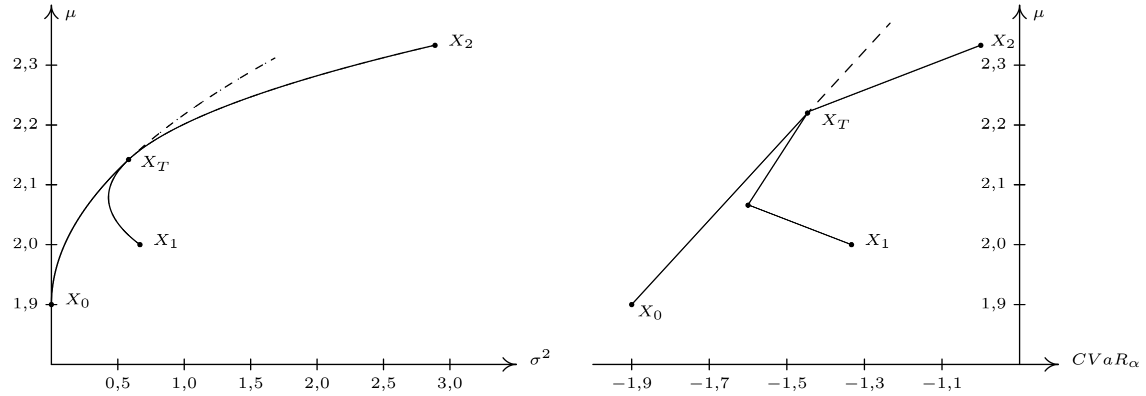


Figure 2: $(\mu, \sigma^2)_2$- versus $(\mu, CVaR_\alpha)_2$-boundary with a risk free and two risky assets

Example 3.8. Let $X_0 = 1{,}9$ be the return of the risk free asset, which is added to the already existing risky assets X_1 and X_2.

Figure 2 shows the $(\mu, \sigma^2)_2$-efficient frontier, which is a strictly concave curve between X_0 and X_{T,σ^2}. The tangency portfolio $\gamma_{T,\sigma^2} = 0{,}5731$ is characterized by the state-dependent returns $X_T = (2{,}28; 1{,}15; 3)'$.

Further, figure 2 shows the $(\mu, CVaR_\alpha)_2$-efficient frontier as a straight line between X_0 and $X_{T,CVaR_\alpha}$. The tangency portfolio $\gamma_{T,CVaR_\alpha} = 0{,}8$ with $X_{T,CVaR_\alpha} = (1{,}6; 1{,}6; 3)'$ is characterized by having at least two identical state-dependent realizations.

3.2. Normal distribution approach

3.2.1. Spectral risk measures under normal distribution

Different from our more general state preference approach, the relevant literature so far has been focused on normally distributed returns (e.g., Alexander/Baptista (2002), Alexander/Baptista (2004), Alexander/Baptista (2006a), De Giorgi (2002), Deng et al. (2009)). This literature in fact analyzes the (μ, ρ_ϕ)-efficient frontiers, but does not explicitly refer to their shape, which is in the center of our contribution. We thus address this open issue in our two-asset framework by now assuming multivariate normally distributed returns. In a way different from the previous literature that finds that the (μ, ρ)-efficient frontiers (almost) coincide, we will find fundamental differences with respect to their shape, and, as a consequence, in the succeeding choice of optimal portfolios. Additionally, the normal distribution approach serves as an example for a continuous state space.

Under the assumption of normally distributed returns, we can rewrite spectral risk measures as

$$\rho_\phi(X) = -E(X) + \rho_\phi(X_N) \cdot \sqrt{Var(X)}, X_N \sim N(0; 1), \tag{27}$$

i.e. as a linear combination of negative expected return and standard deviation (Adam et al. (2008), appendix B, and more general Bertsimas et al. (2004), proposition 1).

Conditional Value-at-Risk in this case becomes

$$\rho_\phi(X) = -E(X) + \alpha^{-1} \cdot n(N^{-1}(\alpha)) \cdot \sqrt{Var(X)}, \tag{28}$$

where n denotes the density function and N^{-1} is the inverse of the standard normal distribution (Alexander/Baptista (2004), p. 1262).

3.2.2. Two risky assets

For variance the results coincide with those from section 3.1.2 (see proposition 3.3).

Next, we analyze (μ, ρ_ϕ)-preferences. By only considering two risky assets, the complete set of alternatives $X_\gamma, \gamma \in [0,1]$ belongs to the $(\mu, \rho_\phi)_1$-boundary. We obtain the $(\mu, \rho_\phi)_1$-boundary by solving the portfolio's expected return for the proportion γ and plugging it into its spectral risk:

$$\rho_\phi(X_\gamma) = -E(X_\gamma) + \rho_\phi(X_N) \cdot \sqrt{\left(\frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)}\right)^2 \cdot a + 2 \cdot \frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)} \cdot b + c}. \tag{29}$$

The second term (stand-alone) constitutes a hyperbola that opens to the right. The additional first term leads to a distortion of the hyperbola, as any risk-return-combination is shifted horizontally by its expected return. The $(\mu, \rho_\phi)_1$-efficient frontier is given by the upper branch of the distorted hyperbola starting from the minimum-spectral risk-portfolio (see figure 3).

Proposition 3.9. *Let $\mathcal{X}$ as in Setting 1. Then the following holds:*

1. *The minimum-spectral risk-portfolio is given by*

$$\gamma_{MSP} = \max\left\{\min\left\{\gamma_{GMSP}; 1\right\}; 0\right\}, \tag{30}$$
$$\gamma_{GMSP} = -\frac{b}{a} - \sqrt{\frac{b^2}{a^2} - \frac{b^2 - t^2 \cdot c}{a \cdot (a - t^2)}}, t^2 = \frac{(E(X_1) - E(X_2))^2}{(\rho_\phi(X_N))^2}.$$

2. *The $(\mu, \rho_\phi)_1$-efficient frontier contains all portfolios $\gamma \in [0, \gamma_{MSP}]$ and lies on a strictly concave curve for $corr(X_1, X_2) \in (-1, 1)$.*

3. *For comonotonic X_1 and X_2, i.e. for $corr(X_1, X_2) = 1$, the $(\mu, \rho_\phi)_1$-efficient frontier lies on a straight line between X_1 and X_2.*

4. *For contramonotonic X_1 and X_2, i.e. for $corr(X_1, X_2) = -1$, the $(\mu, \rho_\phi)_1$-efficient frontier lies on a straight line between the minimum-spectral risk-portfolio and X_2.*

See the appendix for the proof. Strict concavity of the $(\mu, \rho_\phi)_1$-efficient frontier for $corr(X_1, X_2) \in (-1, 1)$ now results from the assumption of continuous and normally distributed random variables that leads to an infinite number of comonotonic subsets of alternatives. This can be seen as a special case of the (discrete) state preference approach for $k \to \infty$. The fundamental difference to variance lies in the fact that the $(\mu, \rho_\phi)_1$-boundary is a *hyperbola* that opens to the right, whereas the $(\mu, \sigma^2)_1$-boundary is a *parabola* that opens to the right. For comonotonicity and contramonotonicity, the $(\mu, \rho_\phi)_1$-efficient frontier reduces to a straight line. The reason is that for normally distributed returns comonotonicity and a perfect positive correlation are equivalent. By contrast, the $(\mu, \sigma^2)_1$-efficient frontier remains a strictly concave curve. Note that for comonotonicity and contramonotonicity, and for $\rho_\phi(X_N) \to \infty$, the minimum-spectral risk-portfolio and the minimum-variance-portfolio coincide.

From propositions 3.3 and 3.9 we can conclude as follows.

Corollary 3.10. *Let $\mathcal{X}$ as in Setting 1. Then the following holds:*

1. $\gamma_{MVP} \geq \gamma_{MSP}$.
2. *The minimum-spectral risk-portfolio is $(\mu, \sigma^2)_1$-efficient and the $(\mu, \rho_\phi)_1$-efficient frontier is a subset of the $(\mu, \sigma^2)_1$-efficient frontier.*

The result preserves for m risky assets: A portfolio belongs to the $(\mu, \rho_\phi)_1$-boundary if and only if it belongs to the $(\mu, \sigma^2)_1$-boundary. If the minimum-spectral risk-portfolio exists, it is $(\mu, \sigma^2)_1$-efficient and lies above the minimum-variance-portfolio. The $(\mu, \rho_\phi)_1$-efficient frontier thus is a subset of the $(\mu, \sigma^2)_1$-efficient frontier.[16]

We give the following example for numerical demonstration.

Example 3.11. An investor can split her initial wealth between two risky assets with multivariate normally distributed returns X_1 and X_2 with $E(X_1) = 2$, $E(X_2) = 2{,}34$, $Var(X_1) = 0{,}67$, $Var(X_2) = 0{,}34$, and $corr(X_1, X_2) = -0{,}24$. (The parameters coincide with those from example 3.5.)

Figure 3 shows the $(\mu, \sigma^2)_1$-boundary, which coincides with figure 1. The minimum-variance-portfolio is given by $\gamma_{MVP} = 0{,}7631$, and the $(\mu, \sigma^2)_1$-efficient frontier contains all portfolios $\gamma \in [0; 0{,}7631]$.

Further, figure 3 shows the $(\mu, CVaR_\alpha)_1$-boundary. The minimum-CVaR-portfolio is given by $\gamma_{MCVaRP} = 0{,}6969$. The $(\mu, CVaR_\alpha)_1$-efficient frontier contains all portfolios $\gamma \in [0; 0{,}6969]$. The $(\mu, CVaR_\alpha)_1$-efficient frontier thus is a subset of the $(\mu, \sigma^2)_1$-efficient frontier.

[16] The proofs are given by Alexander/Baptista (2002), section 2.1 and Alexander/Baptista (2004), section 2.2.2 for Conditional Value-at-Risk. We can easily extend them to the class of spectral risk measures, as they only require a representation of the risk measure of the form (27) with $\rho_\phi(X_N) \geq 0$, which is satisfied by any spectral risk measure. In our two-asset framework, the existence of the minimum-spectral risk-portfolio is guaranteed by the assumption of (μ, ρ_ϕ)-efficient (basic) assets.

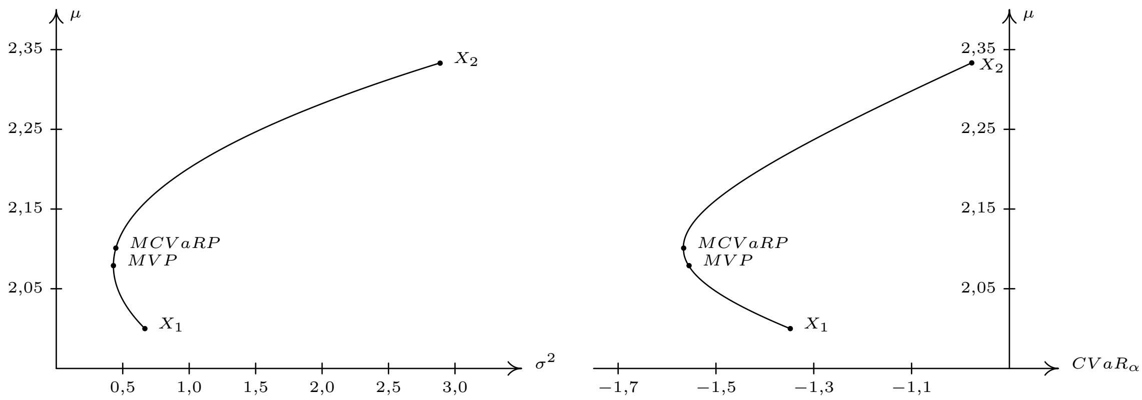


Figure 3: $(\mu, \sigma^2)_1$- versus $(\mu, CVaR_\alpha)_1$-boundary with two risky assets under normal distribution

3.2.3. One risk free and two risky assets

Again, we will consider the risk free asset X_0 in combination with only one risky asset $X_{\bar{\gamma}}$ first, i.e. the set of alternatives reads $X_{\beta,\bar{\gamma}} = \beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0, \beta \in [0,1]$. Afterwards, we will treat the risky asset $X_{\bar{\gamma}}$ as $(\mu, \rho)_1$-efficient portfolio composed of the two risky (basic) assets.

Variance provides us with the same results as in section 3.1.3 (see proposition 3.6).

Next, we turn to (μ, ρ_ϕ)-preferences. We obtain the $(\mu, \rho_\phi)_2$-boundary based on the comonotonic set of alternatives $X_{\beta,\bar{\gamma}}, \beta \in [0,1]$ by solving the portfolio's spectral risk for the proportion β as

$$\begin{aligned} \rho_\phi(X_{\beta,\bar{\gamma}}) &= -(\beta \cdot (E(X_{\bar{\gamma}}) - X_0) + X_0) + \beta \cdot \rho_\phi(X_N) \cdot \sqrt{Var(X_{\bar{\gamma}})} \Leftrightarrow \\ \beta &= \frac{\rho_\phi(X_{\beta,\bar{\gamma}}) + X_0}{\rho_\phi(X_N) \cdot \sqrt{Var(X_{\bar{\gamma}})} - (E(X_{\bar{\gamma}}) - X_0)} \end{aligned} \tag{31}$$

and plugging it into the portfolio's expected return:

$$\begin{aligned} E(X_{\beta,\bar{\gamma}}) =& \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_N) \cdot \sqrt{Var(X_{\bar{\gamma}})} - (E(X_{\bar{\gamma}}) - X_0)} \cdot \rho_\phi(X_{\beta,\bar{\gamma}}) + \\ & \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_N) \cdot \sqrt{Var(X_{\bar{\gamma}})} - (E(X_{\bar{\gamma}}) - X_0)} \cdot X_0 + X_0. \end{aligned} \tag{32}$$

The portfolio's expected return and its spectral risk are again in a linear relationship. Note that (32) is only a special case of (26) for normally distributed returns.

Generally, any $(\mu, \rho_\phi)_1$-efficient portfolio can serve as risky asset in (32). The $(\mu, \rho_\phi)_2$-efficient frontier thus is a straight line between the risk free asset and the tangency portfolio X_{T,ρ_ϕ}. Again, the tangency portfolio is characterized by having the maximum slope within the portfolios of the $(\mu, \rho_\phi)_1$-efficient frontier (see figure 4).

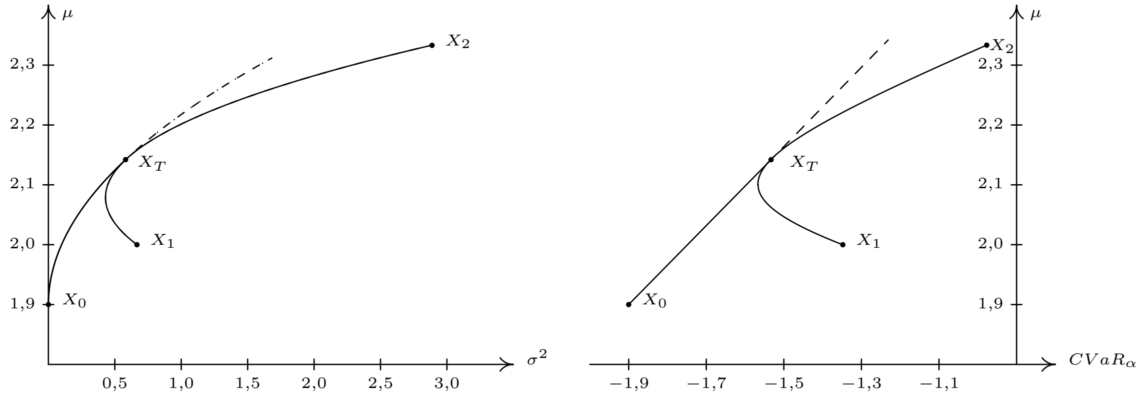


Figure 4: $(\mu, \sigma^2)_2$- versus $(\mu, CVaR_\alpha)_2$-boundary with a risk free and two risky assets

Proposition 3.12. *Let $\mathcal{X}$ as in Setting 2 and $-\rho_\phi(X_{GMSP}) < X_0 < E(X_A) = E(X_{GMVP})$. Then the following holds:*

1. *The $(\mu, \rho_\phi)_2$-efficient frontier lies on a straight line between the risk free asset and the tangency portfolio.*
2. *The tangency portfolio γ_{T,ρ_ϕ} is given by (24), i.e. the tangency portfolios under variance and spectral risk measures coincide.*

See the appendix for the proof. TOBIN's separation theorem is still valid. Note that for any given parameters $E(X_1), E(X_2), Var(X_1), Var(X_2)$, and $corr(X_1, X_2) \in (-1, 1)$, there exists a corresponding value X_0, so that now any $(\mu, \sigma^2)_1$-efficient portfolio can serve as tangency portfolio, i.e. $\gamma_{T,\rho_\phi} \in [0, 1]$.

The above result preserves for m risky assets: Under additional existence conditions, the tangency portfolios under variance and spectral risk measures coincide.[17] Especially, the $(\mu, \rho_\phi)_2$-efficient frontier according to (32) is still a straight line between the risk free asset and the tangency portfolio.

In the following example, we add the risk free asset to example 3.11.

Example 3.13. Let $X_0 = 1{,}9$ be the return of the risk free asset, which is added to the two risky assets from example 3.11.

Figure 4 shows the $(\mu, \sigma^2)_2$-efficient frontier, which is given by a strictly concave curve between the risk free asset and the tangency portfolio $\gamma_{T,\sigma^2} = 0{,}5731$.

Further, figure 4 shows the $(\mu, CVaR_\alpha)_2$-efficient frontier, which is given by a straight line between the risk free asset and tangency portfolio $\gamma_{T,\rho_\phi} = 0{,}5731$.

Table 1 summarizes the main results from section 3. We find that whereas strict concavity prevails under variance, (piecewise) linearity dominates under spectral risk measures. This

[17] See DE GIORGI (2002), p. 13 and corollary 5.1. The existence conditions in our two-asset framework are satisfied by the assumption of (μ, ρ_ϕ)-efficient (basic) assets.

		state preference approach variance	state preference approach spectral rm	normal distribution approach variance	normal distribution approach spectral rm
entirely risky	general	parabola	piecewise linear	parabola	hyperbola
		strictly concave	piecewise linear	strictly concave	strictly concave
	comonotonic	parabola	straight line	parabola	straight line
		strictly concave	linear	strictly concave	linear
risky and risk free	general	square-root	straight line	square-root	straight line
		strictly concave	linear	strictly concave	linear
	tang. portf.	all	corners	all	all

Table 1: Summary

linearity has not been made explicit in the previous literature, and it results from the regulatory concept of diversification underlying spectral risk measures that regards the dependence structure as the only source for positive diversification benefits ("risk"). The following analysis of the choice of optimal portfolios ("decision") will show that linearity yields restrictive portfolio structures that are contrary to our initial intuition.

4. Optimal portfolios

4.1. Indifference curves

We now turn to the choice of optimal portfolios, since criticism on its results also questions the rationality of the preceding determination of (μ, ρ)-efficient frontiers.

Definition 4.1. *A portfolio is said to be optimal with respect to a utility function π if it maximizes π along a set of alternatives $\mathcal{X}$.*

For the choice of optimal portfolios, we will apply linear reward-risk-models using expectation as reward measure, and variance and spectral risk measures as risk components. Clearly, the optimal portfolios are located where the induced indifference curves are tangent to the (μ, ρ)-efficient frontiers.

The hybrid model

$$\pi(X) = E(X) - \frac{\lambda}{2} \cdot Var(X), \lambda \geq 0 \tag{33}$$

will be applied to the (μ, σ^2)-efficient frontiers, and will serve as a rational benchmark. This utility function is usually justified with expected utility theory under normally distributed returns and an exponential utility function (e.g., Bamberg (1986), p. 20). These assumptions are well-established in portfolio theory due to their striking analytical advantages (e.g., Bamberg (1986), Lintner (1969), Lintner (1970), Sentana (2003)). The induced indifference curves are linearly increasing functions with slope $\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{\lambda}{2} \geq 0$.

Spectral utility functions are of the form

$$\pi_\phi(X) = (1-\lambda)\cdot E(X) - \lambda\cdot\rho_\phi(X), \lambda\in[0,1] \tag{34}$$

(see section 2.2 above). The induced indifference curves are also linearly increasing functions with slope $\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \lambda/(1-\lambda) \geq 0$. Below we treat λ as the only risk aversion parameter, whereas ϕ is assumed to be given ex-ante.

4.2. The hybrid model and full diversification

As our considerations on the (μ,σ^2)-efficient frontiers did not reveal any difference between the state preference approach and the normal distribution approach, we can discuss them together below.

The $(\mu,\sigma^2)_1$-efficient frontier is located on the strictly concave upper branch of a parabola also for comonotonic risky assets. Its marginal rate of transformation according to (19) for $\gamma\in[0,\gamma_{MVP}]$ is given by

$$\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{E(X_1)-E(X_2)}{2\cdot(\gamma\cdot a+b)} \in \left[\frac{E(X_1)-E(X_2)}{2\cdot b}, \begin{cases}\infty & \text{if } -\frac{b}{a}\leq 1\\ \frac{E(X_1)-E(X_2)}{2\cdot(a+b)} & \text{else}\end{cases}\right). \tag{35}$$

By including the risk free asset, the $(\mu,\sigma^2)_2$-efficient frontier lies on a strictly concave curve. The marginal rate of transformation according to (23) for $\beta\in[0,1]$ reads

$$\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{E(X_{T,\sigma^2})-X_0}{2\cdot Var(X_{T,\sigma^2})\cdot\beta} \in \left[\frac{E(X_{T,\sigma^2})-X_0}{2\cdot Var(X_{T,\sigma^2})},\infty\right). \tag{36}$$

In connection with the linear indifference curves of the hybrid model (33), i.e. a constant marginal rate of substitution, we immediately get the following proposition.

Proposition 4.2. *Let an investor maximize the hybrid model (33) with respect to β and γ in Setting 1 and 2, respectively. Then the following holds:*

$$\gamma^* = \max\left\{\min\left\{\frac{E(X_1)-E(X_2)}{\lambda\cdot a} - \frac{b}{a};1\right\};0\right\}, \tag{37}$$

$$\beta^* = \max\left\{\min\left\{\frac{E(X_{T,\sigma^2})-X_0}{\lambda\cdot Var(X_{T,\sigma^2})};1\right\};0\right\}. \tag{38}$$

Regarding our initial intuition, we can conclude as follows (see figure 5):

- The optimal asset allocation regularly yields diversifying portfolio structures $\beta^*,\gamma^*\in(0,1)$ that reflect the optimal tradeoff between an investor's risk aversion λ and a positive risk premium. Diversification also obtains for comonotonicity, i.e. for comonotonic risky assets, and the risk free asset and the tangency portfolio, respectively. This appears to be

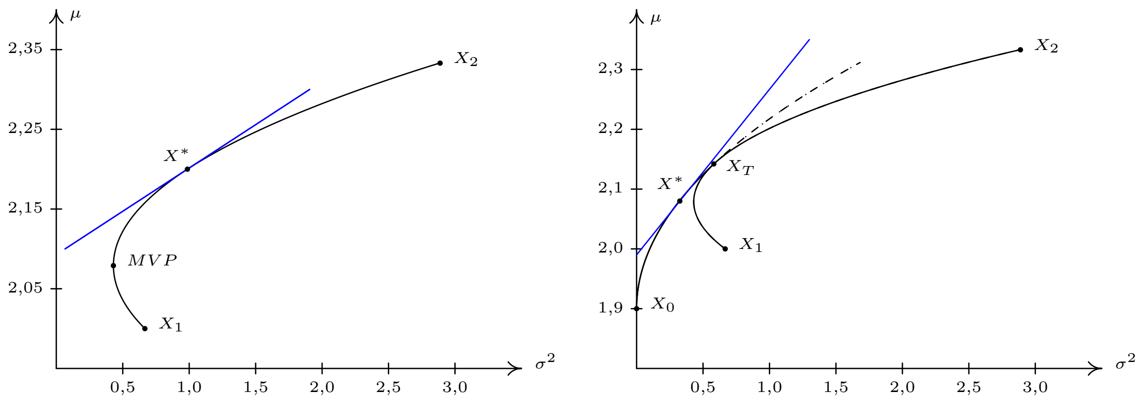


Figure 5: Choice of optimal portfolios with the hybrid model

consistent although there are no diversification benefits from the dependence structure, as an investor may prefer a ((μ, σ^2)-efficient) risk-return profile that lies in the interior of the comonotonic assets' risk-return profiles.

- The optimal asset allocation is continuous in the risk aversion λ. At a marginal increase (decrease) in risk aversion, the investment in the riskier asset (i.e., X_2 in the case of two risky assets, and X_{T,σ^2} in the case of the additional risk free asset) decreases (increases) marginally. For the limiting cases of infinite risk aversion $\lambda \to \infty$ and risk neutrality $\lambda \to 0$, the exclusive investment in the most riskless (i.e., X_{MVP}, or X_0) and in the riskiest asset (i.e., X_2, or X_{T,σ^2}), respectively, results.

- The optimal asset allocation is continuous in the risk premium. At a higher (lower) risk premium, a larger (smaller) proportion is invested riskier (in X_2, or X_{T,σ^2}).

- Any (μ, σ^2)-efficient portfolio can be optimal for an appropriate risk aversion λ.

The hybrid model is consistent with our initial intuition, and we have *full diversification*. Nonetheless, from the perspective of decision theory consistency with expected utility theory holds under only strict assumptions, and is lacking otherwise. Especially, variance only captures investors' true risk preferences under normally distributed or, more general, elliptical returns (e.g., Chamberlain (1983), Owen/Rabinovitch (1983)).

4.3. Spectral utility functions and limited diversification

Under spectral utility functions, we have to differentiate between the (discrete) state preference approach and the (continuous) normal distribution approach.

We start with the two risky assets. In state preference approach, the $(\mu, \rho_\phi)_1$-efficient frontier is located on a concave curve that for comonotonic subsets of alternatives is piecewise linear. For a comonotonic subset of alternatives $X_\gamma, \gamma \in [\gamma_d, \gamma_u]$ we according to (21) obtain the following constant marginal rate of transformation:

$$\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})}. \tag{39}$$

In connection with the linear indifference curves of spectral utility functions, i.e. a constant marginal rate of substitution, we immediately get the following result.[18]

Proposition 4.3. *Let an investor maximize spectral utility functions (34) with respect to γ in Setting 1. Then the following holds:*

$$\gamma^* \in \{0, \gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}, 1\}. \tag{40}$$

Regarding our initial intuition, it holds that (see figure 6):

- Not any $(\mu, \rho_\phi)_1$-efficient portfolio can be optimal, only the boundaries of the comonotonic subsets of alternatives come into consideration. We thus obtain *limited diversification* only.

  For comonotonic risky assets limited diversification attains its maximum. Then the $(\mu, \rho_\phi)_1$-efficient frontier is a straight line between the risky assets, so that, contrary to variance, diversification is never optimal. Instead, the exclusive investment in one of the risky assets obtains. An investor thus cannot prefer a risk-return profile that lies in the interior of the comonotonic risky assets' risk-return profiles, although it is $(\mu, \rho_\phi)_1$-efficient. As comonotonicity only implies $corr(X_1, X_2) \in [0, 1]$, all-or-nothing-decisions may also occur for any $corr(X_1, X_2) \in [0, 1]$.

  Conversely, we observe all-or-nothing-decisions that hold for any dependence structure if there are only two states of the world. In this case, the portfolio $\gamma_{ij,1:1}$ by definition has identical state-dependent returns and thus corresponds to a synthetical risk free asset. Accordingly, an investor either decides for one of the risky assets, or she decides for the synthetical risk free asset.

- The optimal proportion γ^* lacks sensitivity, as it is not continuous in the risk aversion λ and in the risk premium. With an increasing risk aversion and a decreasing risk premium (and vice versa), the same proportions remain optimal until the portfolio jumps to the next corner. This effect is similar to find an optimal solution using the simplex-algorithm, which moves along the edges of the feasible region and jumps in the polyeder's corner positions.

For continuous normal distribution, the $(\mu, \rho_\phi)_1$-efficient frontier for $corr(X_1, X_2) \in (-1, 1)$ is located on a strictly concave curve (upper branch of a hyperbola). Supposedly, we obtain full diversification. This framework, which is common in the relevant literature as yet, thus

[18]Without loss of generality we assume that if the marginal rate of transformation and the marginal rate of substitution coincide, the investor decides for a corner position.

gives the impression that spectral risk measures regularly exhibit full diversification. On the other hand, from our more general state preference approach we know that this kind of full diversification is only a special case of limited diversification for continuous normal distribution. The difference to true full diversification as under variance becomes already clear for comonotonic risky assets, i.e. for $corr(X_1, X_2) = 1$. Then, limited diversification again attains its maximum, as the $(\mu, \rho_\phi)_1$-efficient frontier reduces to a straight line, and diversification does not occur. Instead, the exclusive investment in one of the risky assets obtains.

The case of an additional risk free asset brings us to our main result. Both within the state preference approach, and within the normal distribution approach, the $(\mu, \rho_\phi)_2$-efficient frontier is a straight line between the risk free asset and the tangency portfolio. Its constant marginal rate of transformation according to (26) and (32) is given by

$$\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{E(X_{T,\rho_\phi}) - X_0}{\rho_\phi(X_{T,\rho_\phi}) - \rho_\phi(X_0)} \geq 0. \tag{41}$$

In connection with the linear indifference curves of spectral utility functions, i.e. a constant marginal rate of substitution, we can immediately conclude as follows.

Proposition 4.4. *Let an investor maximize spectral utility functions (34) with respect to β in Setting 2. Then the following holds:*

$$\beta^* = \begin{cases} 0 & \textit{if } \frac{E(X_{T,\rho_\phi})-X_0}{\rho_\phi(X_{T,\rho_\phi})-\rho_\phi(X_0)} \geq \frac{\lambda}{1-\lambda} \\ 1 & \textit{else} \end{cases}. \tag{42}$$

Either the exclusive investment in the risk free asset or in the tangency portfolio obtain as optimal solutions (see figure 6). Spectral utility functions already restrict diversification on the elementary level "risk free versus risky", and we have maximum limited diversification. On the one hand, this can lead to the contra-intuitive situation that a risk averse investor decides for an exclusive investment in the tangency portfolio, although the risk free asset is available. Conversely, a risk averse investor may decide for the exclusive investment in the risk free asset, although the tangency portfolio offers a positive risk premium. Regarding our initial intuition, it holds that:

- The optimal proportion β^* lacks sensitivity, as it is not continuous in the risk aversion λ and in the risk premium. Up to a certain degree of risk aversion and risk premium, the exclusive investment in the tangency portfolio is optimal. Subsequently, the optimum moves towards the exclusive risk free investment. The underlying concept of risk aversion is consistent insofar, as a more risk averse investor reaches this point earlier, i.e. at a higher risk premium (see (42)). This all-or-nothing-decision, however, counteracts the tradeoff between an investor's risk aversion and a positive risk premium, which is always

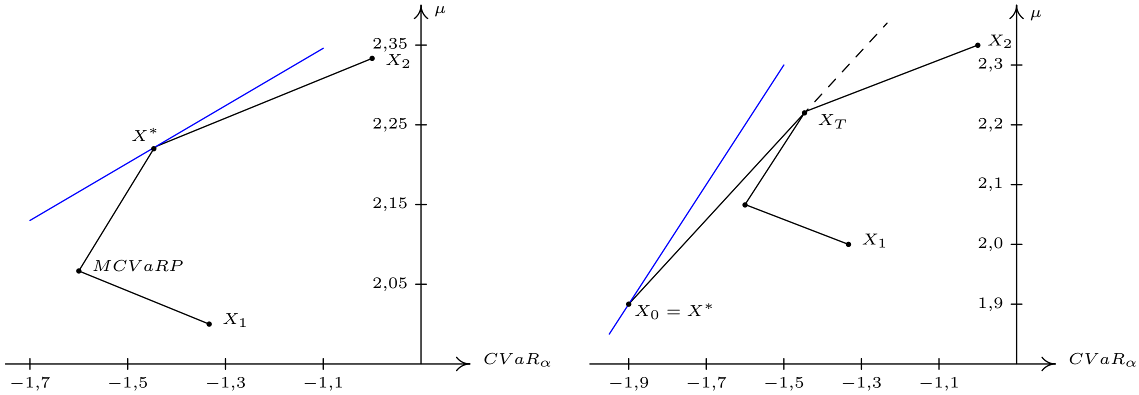


Figure 6: Choice of optimal portfolios with spectral utility functions

solved in one of the corners $\beta^* \in \{0, 1\}$. Whether an investor decides for the risk free asset or for the tangency portfolio depends on the assets' characteristics and on the risk aversion. Generally, both cases are possible.

- $(\mu, \rho_\phi)_2$-efficient portfolios are excluded from being optimal.

We finally show that the all-or-nothing-decision is robust to m risky assets. In this case, the investor solves

$$\max_{\beta,\gamma_1,\ldots,\gamma_m \in [0,1]} \pi_\phi \left(\beta \cdot \sum_{i=1}^{m} \gamma_i \cdot X_i + (1-\beta) \cdot X_0 \right), \beta + \sum_{i=1}^{m} \gamma_i = 1, \tag{43}$$

which due to translation invariance and positive homogeneity yields the first order conditions

$$\frac{\partial \pi_\phi(X_{\beta,\gamma})}{\partial \beta} = \pi_\phi \left(\sum_{i=1}^{m} \gamma_i \cdot X_i \right) - X_0, \tag{44}$$

$$\frac{\partial \pi_\phi(X_{\beta,\gamma})}{\partial \gamma_i} = \beta \cdot \frac{\partial \pi_\phi \left(\sum_{i=1}^{m} \gamma_i \cdot X_i \right)}{\partial \gamma_i}. \tag{45}$$

According to (45), the investor first chooses an optimal portfolio composed entirely of risky assets. Second, she according to (44) either decides for an exclusive investment in this risky (tangency) portfolio or in the risk free asset.

5. Discussion

Our findings on limited diversification within the integrating framework provide strong implications for the application of spectral risk measures and spectral utility functions. If an investor does not agree with either investing exclusively risk free or investing exclusively risky, she must not use spectral utility functions. As a consequence of the integrating framework, the portfolio selection then must not be based on (μ, ρ_ϕ)-efficient frontiers either. Therefore, the (μ, ρ_ϕ)-framework lacks a foundation, at least from the perspective of decision theory. This holds especially, as this framework cannot be motivated as a regulatory approach either. Additionally, from experimental studies it is well-known that investors regularly diversify between a risk free and a risky asset (e.g., Benartzi/Thaler (1999), Levy (1994), Rapoport (1984), and Haisley et al. (2010) in a recent study). We thus conclude that the use of spectral risk measures for portfolio selection appears inappropriate from both a theoretical and an empirical perspective. This view is already supported by Markowitz (1952), who notes that "diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim" (p. 77).

From a more formal perspective, the all-or-nothing-decision may seem surprising at first sight, as spectral utility functions satisfy the property of superadditivity that is motivated by diversificational arguments. However, in the case of "risk free versus risky", superadditivity is overlaid by a completely comonotonic set of alternatives in connection with comonotonic additivity of spectral utility functions. Therefore, the maximization of spectral utility functions due to (6) reduces to a simple maximization of expected return, which clearly brings out corner solutions. If an investor does not agree with an all-or-nothing-decision, she must not use a utility function that exhibits the property of comonotonic additivity. This in turn prohibits the determination of (μ, ρ_ϕ)-efficient frontiers and the choice of optimal portfolios using spectral utility functions, which are comonotonic additive. The differences between variance and spectral risk measures from that point of view are minor but important: Whereas variance is strictly convex on $\mathcal{X}$ and exhibits full diversification, spectral risk measures are convex but piecewise linear on $\mathcal{X}$ and show limited diversification only.

In a weakened form the same argument on limited diversification applies for risky assets. From the perspective of decision theory, we find no economic reason for excluding $(\mu, \rho_\phi)_1$-efficient portfolios from being optimal, only because they belong to a comonotonic subset of alternatives. This only means that both expected return and spectral risk increase linearly. It might be well possible, and economically plausible, that an investor prefers a $(\mu, \rho_\phi)_1$-efficient risk-return profile that lies in the interior of a comonotonic subset of alternatives.

The restrictive all-or-nothing-decisions results from simply adopting the properties of spectral risk measures for purposes of optimal decision making in the context of portfolio selection. In the original setting of solvency capital assessment ("risk"), they induce a conclusive concept of diversification that exclusively relates to the dependence structure between the assets. Portfolio selection ("decision"), by contrast, demands a concept of diversification that is based on the optimal tradeoff between risk and return, and that is only indirectly affected by the dependence structure.

In a way, our findings are in contrast to those of the previous literature. Alexander/Baptista (2004) and Alexander/Baptista (2006a) argue that an additional Conditional Value-at-Risk constraint in the absence of the risk free asset can lead to "perverse situations" in that it is more likely that investors select portfolios with a higher standard deviation. If the risk free asset exists, these adverse effects disappear. Similarly, De Giorgi

(2002) shows that under the assumption of normally distributed returns the $(\mu, \rho)_2$-efficient frontiers under variance and Conditional Value-at-Risk coincide in the presence of the risk free asset, whereas they differ in the absence of the risk free asset. However, in both cases the argumentation is based on (μ, ρ)-efficient frontiers only, and does not consider the succeeding choice of optimal portfolios as we do. In doing so, we find perverse situations in the presence of the risk free asset in that diversification does not occur. By refraining from the risk free asset, diversification, although limited, becomes more likely in our integrating framework.

While in the present literature on spectral risk measures and spectral utility functions these results have been overlooked so far, our criticism is well-known from the dual theory of choice. Already YAARI (1987), section 6, notes that the dual theory of choice tends to all-or-nothing-decisions (plunging) instead of diversification. On the other hand, HADAR/SEO (1995) give conditions under which dual investors diversify between two (or more) risky assets. Nonetheless, they do not consider the risk free asset, and they do not refer to the portfolio structure itself as we do.

The problem of corner solutions induced by spectral risk measures and spectral utility functions is also incorporated in numerous other economic applications. Especially, if a decision problem depends on only one random variable, the set of alternatives is usually comonotonic, and thus the linearity of spectral utility functions prevails. Anyhow, the corner solutions often remain hidden at first sight due to the nonlinearity of the sets of alternatives, but can be recovered for special cases. We only sketch some examples:

- For deductible insurance contracts, the set of alternatives is comonotonic but nonlinear (DOHERTY/EECKHOUDT (1995), section 3.1.2). For a binary loss random variable, the set of alternatives becomes linear, so that under spectral utility functions either zero or full coverage obtain as optimal solutions. More general, for discrete loss random variables only the existing realizations come into consideration as optimal deductibles.

- With proportional insurance contracts, the linearity of the set of alternatives holds irrespective of the underlying distribution of the loss random variable. Therefore, either zero or full coverage is optimal under spectral utility functions (DOHERTY/EECKHOUDT (1995), section 3.1.1).

- In production theory with uncertain demand (newsvendor model, see KHOUJA (1999) for an extensive overview) the set of alternatives again is comonotonic and nonlinear, but it becomes linear for binary demand random variables. This implies that under spectral utility functions the optimal order quantity corresponds either to the minimal or to the maximal demand (e.g., CHAHAR/TAAFFE (2009) and TAAFFE ET AL. (2008) for all-or-nothing (AON) demand models). Likewise, for discrete demand random variables only the existing realizations come into consideration as optimal order quantities.

- In production theory under price uncertainty (see Sandmo (1971), section I, for the basic model and results in expected utility framework) the set of alternatives is linear irrespective of the underlying price distribution. The optimal output level under spectral utility functions thus is either zero or the capacity limit.

6. Conclusion

The paper starts with the observation that the literature on portfolio theory with Conditional Value-at-Risk and spectral risk measures so far lacks an integrating framework, as the choice of optimal portfolios is not subject to considerations. We thus modify the prevalent portfolio selection approaches in two respects.

First, we refrain from assuming normally distributed returns that by definition yield similar portfolio structures compared to the (μ, σ^2)-approach. Instead, we introduce the state preference approach that does not require any initial distribution. We find that whereas under variance the efficient frontiers are strictly concave in any case, piecewise linearity obtains under spectral risk measures. Especially, if the risk free asset exists, the $(\mu, \rho_\phi)_2$-efficient frontier is linear.

Second, we also consider the choice of optimal portfolios within an integrating framework for the first time. We show that any spectral utility function consists of a convex combination of expectation and a (negative) spectral risk measure. By confronting these spectral utility functions with the (μ, ρ_ϕ)-efficient frontiers, we find limited diversification only. In the case of risky assets, only the boundaries of the comonotonic subsets of alternatives come into consideration as optimal portfolios, although the interior points are $(\mu, \rho_\phi)_1$-efficient (limited diversification). If the risk free asset exists, limited diversification attains its maximum in that diversification is never optimal. Either the investment in the risk free asset, or the investment in the tangency portfolio obtains. By contrast, under (μ, σ^2)-preferences with the hybrid model we find full diversification in any case. As diversification is a key issue in portfolio theory, the use of spectral risk measures appears inappropriate from both a theoretical and an empirical perspective.

The reason is that spectral risk measures have originally been introduced for the assessment of solvency capital ("risk"). The underlying regulatory concept of diversification regards the dependence structure between the assets as the only source for positive diversification benefits. For the special case of perfect positive dependence (comonotonicity) the diversification benefit is zero, which is an adequate requirement for the assessment of solvency capital. Portfolio selection ("decision"), by contrast, demands a different concept of diversification that is based on the optimal tradeoff between risk and reward, and that is only indirectly affected by the dependence structure between the assets. The incompatibility of these conflicting concepts of diversification brings out the limited diversification.

In formal terms, the concept of diversification underlying spectral risk measures and spectral utility functions is determined jointly by the properties of superadditivity *and* comonotonic additivity. The relevant literature, by contrast, is focused on subadditivity only, and omits to consider comonotonic additivity, e.g., "sub-additivity is an essential property also in portfolio-optimization problems" (Acerbi/Tasche (2002a), p. 381). We indeed agree in that subadditivity and superadditivity, respectively, are essential properties for the assessment of solvency capital and for portfolio selection. However, we show that comonotonic additivity is an essential property for the assessment of regulatory capital only, and leads to paradoxical results if applied for portfolio selection.

Notwithstanding these findings, we have no doubt that an axiomatic foundation is useful to avoid mathematical and contextual inconsistencies, and preserves investors from choosing a risk measure or utility function somewhat ad hoc. On the other hand, axiomatic approaches are by no means a universal concept that can be applied to any decision context regardless of the original context they have been developed for. Our view contrasts with Acerbi/Tasche (2002a), who "clearly state that in our opinion speaking of non-coherent (and non-spectral, the author) measures of risk is (..) useless and dangerous. In our language, the adjective coherent is simply redundant." (p. 380). On the other hand, we show that claiming that axiomatic approaches are per se superior without taking account of the consequences from both an economic and a decision-theoretic perspective does not suffice in any way. As a more appropriate alternative to spectral risk measures, and as an agenda for future research, we thus propose convex risk measures, which do not require for comonotonic additivity (e.g., Föllmer/Schied (2002), Frittelli/Gianin (2002)).

A. Proof of proposition 3.3

For a portfolio's variance we have

$$Var(X_\gamma) = \gamma^2 \cdot a + 2 \cdot \gamma \cdot b + c. \tag{46}$$

Differentiating this expression for γ yields

$$\frac{\mathrm{d}Var(X_\gamma)}{\mathrm{d}\gamma} = 2 \cdot \gamma_{GMVP} \cdot a + 2 \cdot b \stackrel{!}{=} 0 \Leftrightarrow \gamma_{GMVP} = -\frac{b}{a}. \tag{47}$$

The $(\mu, \sigma^2)_1$-efficient frontier is a strictly concave curve for any correlation coefficient, $corr(X_1, X_2) \in [-1, 1]$, as $\frac{\mathrm{d}^2 Var(X_\gamma)}{\mathrm{d}\gamma^2} = 2 \cdot a > 0$ for any correlation coefficient, $corr(X_1, X_2) \in [-1, 1]$. Therefore, strict concavity also holds for comonotonicity.

B. Proof of proposition 3.6

For the proof, we maximize the sharpe-ratio. The tangency portfolio thus is given by

$$\gamma_{T,\sigma^2} = \arg\max_{\gamma\in[0,1]} \frac{\gamma \cdot E(X_1) + (1-\gamma)\cdot E(X_2) - X_0}{\sqrt{\gamma^2 \cdot a + 2\cdot\gamma\cdot b + c}}. \tag{48}$$

Differentiating the right hand side for γ provides us with the first order condition

$$\begin{aligned}&(E(X_1) - E(X_2))\cdot\sqrt{\gamma^2\cdot a + 2\cdot\gamma\cdot b + c} -\\ &((\gamma\cdot E(X_1) + (1-\gamma)\cdot E(X_2)) - X_0)\cdot\frac{\gamma\cdot a + b}{\sqrt{\gamma^2\cdot a + 2\cdot\gamma\cdot b + c}} \overset{!}{=} 0,\end{aligned} \tag{49}$$

which can be solved for γ as

$$\gamma = \frac{(E(X_2) - X_0)\cdot b - (E(X_1) - E(X_2))\cdot c}{(E(X_1) - E(X_2))\cdot b - (E(X_2) - X_0)\cdot a}. \tag{50}$$

As the $(\mu,\sigma)_1$-boundary is symmetric around the minimum-variance-portfolio, we attain a maximum for $X_0 < E(X_{GMVP})$.

C. Proof of proposition 3.9

We obtain the minimum-spectral risk-portfolio by differentiating the portfolio's spectral risk

$$\rho_\phi(X_\gamma) = -(\gamma\cdot E(X_1) + (1-\gamma)\cdot E(X_2)) + \rho_\phi(X_N)\cdot\sqrt{\gamma^2\cdot a + 2\cdot\gamma\cdot b + c} \tag{51}$$

for γ:

$$\frac{\mathrm{d}\rho_\phi(X_\gamma)}{\mathrm{d}\gamma} = -(E(X_1) - E(X_2)) + \rho_\phi(X_N)\cdot\frac{\gamma\cdot a + b}{\sqrt{\gamma^2\cdot a + 2\cdot\gamma\cdot b + c}} \overset{!}{=} 0. \tag{52}$$

Solving this equation for γ yields

$$\gamma = -\frac{b}{a} - \sqrt{\frac{b^2}{a^2} - \frac{b^2 - t^2\cdot c}{a\cdot(a - t^2)}}, t^2 = \frac{(E(X_1) - E(X_2))^2}{(\rho_\phi(X_N))^2}. \tag{53}$$

For the existence of the minimum-spectral risk-portfolio, the condition

$$a \geq t^2 \Leftrightarrow \rho_\phi(X_N) \geq \frac{|E(X_1) - E(X_2)|}{\sqrt{a}} \tag{54}$$

must hold. As we assume the (basic) assets to be (μ,ρ_ϕ)-efficient we have

$$\rho_\phi(X_N) \geq \frac{E(X_2) - E(X_1)}{\sqrt{Var(X_2)} - \sqrt{Var(X_1)}}, \tag{55}$$

which in turn implies equation (54). Finally, for the second derivate we attain

$$\begin{aligned}
\frac{\mathrm{d}^2\rho_\phi(X_\gamma)}{\mathrm{d}\gamma^2} &= \rho_\phi(X_N)\cdot\frac{a\cdot\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}-\frac{(\gamma\cdot a+b)^2}{\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}}}{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}\\
&= \rho_\phi(X_N)\cdot\frac{\sqrt{Var(X_1)\cdot Var(X_2)}\cdot(1-corr(X_1,X_2)^2)}{(\gamma^2\cdot a+2\cdot\gamma\cdot b+c)^{\frac{3}{2}}}\\
&\begin{cases} >0 & corr(X_1,X_2)\in(-1,1)\\ =0 & \text{else}\end{cases}.
\end{aligned} \tag{56}$$

D. Proof of proposition 3.12

The tangency portfolio is given by

$$\gamma_{T,\rho_\phi} = \underset{\gamma\in[0,1]}{\arg\max}\frac{\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2)-X_0}{-(\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2))+\rho_\phi(X_N)\cdot\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}+X_0}. \tag{57}$$

Differentiating the right hand side for γ provides us with the first order condition

$$\begin{aligned}
&(E(X_1)-E(X_2))\cdot(-(\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2))+\rho_\phi(X_N)\cdot\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}+X_0)\\
&-((\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2))-X_0)\cdot\\
&\left(-(E(X_1)-\cdot E(X_2))+\rho_\phi(X_N)\cdot\frac{\gamma\cdot a+b}{\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}}\right)\overset{!}{=}0.
\end{aligned} \tag{58}$$

As we can extract the difference

$$\begin{aligned}
&(E(X_1)-E(X_2))\cdot(-((\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2))-X_0))-\\
&(E(X_1)-E(X_2))\cdot(-((\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2))-X_0))=0
\end{aligned} \tag{59}$$

from the first and the second summand, the first order condition reduces to

$$\begin{aligned}
&(E(X_1)-E(X_2))\cdot\rho_\phi(X_N)\cdot\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}-\\
&((\gamma\cdot E(X_1)+(1-\gamma)\cdot E(X_2))-X_0)\cdot\rho_\phi(X_N)\cdot\frac{\gamma\cdot a+b}{\sqrt{\gamma^2\cdot a+2\cdot\gamma\cdot b+c}}\overset{!}{=}0,
\end{aligned} \tag{60}$$

which is equivalent to the first order condition (49) and thus proves the assertion.

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