Simple Portfolio Choice for HARA Investors [1]

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HARA-utility investors allocate their money to a risk-free fund and to a risky fund (two fund separation). The paper shows that under weak conditions, the structure of the risky fund and the amount invested in this fund can be approximated by those derived from a utility function with low constant relative risk aversion, without material effects on the certainty equivalent of the portfolio payoff. The approximation is of high quality if the investor's level of relative risk aversion is higher than that used for approximation and if approximate arbitrage opportunities do not exist.

Keywords:	HARA-utility, portfolio choice, certainty equivalent, approximated choice
JEL-classification:	G10, G11, D81

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1 Introduction

Over the last decades a sophisticated theory of decision making under uncertainty, based on the expected utility paradigm, has been developed. Following the seminal papers by Arrow (1974), Pratt (1964), Rothschild and Stiglitz (1970), Diamond and Stiglitz (1974), many papers investigated optimal decisions and showed how they depend on the utility function. In finance, portfolio choice is perhaps the most important application of expected utility theory. Yet, there is a long discussion whether sophisticated approaches to portfolio choice are really worthwhile. Based on the (μ, σ)-model, many papers discuss the impact of parameter uncertainty. Several papers conclude that simple rules for portfolio choice perform as well as sophisticated rules.

This paper proposes a related result for a wide set of utility functions. It discusses optimal portfolio choice for utility functions with hyperbolic absolute risk aversion (HARA) in the absence of parameter uncertainty. This class of functions with declining absolute risk aversion is heavily used in finance. The purpose of this paper is to show that the optimal portfolio of a HARA-investor can well be approximated by the optimal portfolio derived for constant relative risk aversion ϕ, if the investor's relative risk aversion γ is higher than ϕ and if asset returns rule out approximate arbitrage opportunities. The approximated portfolio is a long position in the optimal portfolio, given relative risk aversion ϕ, plus a risk-free investment. The higher is the relative risk aversion γ of the investor, the less (more) money she invests in the risky portfolio (risk-free asset). We measure the approximation quality by the approximation loss, i.e. the relative increase in initial endowment required to raise the certainty equivalent of the approximated portfolio to that of the optimal portfolio. The approximation loss turns out to be very small in many settings. Moreover, the paper shows that an investor buying stocks and a risk-free asset may well buy the market portfolio or a transformed market portfolio with the fraction of initial wealth invested in the market portfolio being a simple hyperbolic function of her relative risk aversion. Thus, simple rules of asset allocation do a very good job. These results are derived in the absence of uncertainty about asset return parameters. Since this uncertainty also appears to support simple rules for asset allocation, there is double support for simple rules. As a caveat, our results should not be applied to risk management which focuses on tail risks. Our results are based on the certainty equivalent of portfolio payoffs covering the full distribution of payoffs.

The practical relevance of this finding is easily illustrated. A portfolio manager has many different customers investing in different risky funds and the risk-free asset. Their preferences may be characterized by increasing, constant or declining relative risk aversion (RRA) and can be approximated by a HARA-function with an exponent γ clearly above 0. First, the manager derives the portfolio for some low constant RRA ϕ, say, 1. Second, the manager allocates the customer's initial endowment to the same portfolio and the risk-free asset, investing the fraction (ϕ/customer's γ) in that portfolio and the rest in the risk-free asset. Hence, the allocations for different customers only differ by the amount invested in that portfolio and the amount invested risk-free.

This approximation generalizes two fund-separation derived by Cass and Stiglitz (1970). They proved this separation for any HARA-function given the exponent ϕ. This paper argues that the same risky portfolio may be used for all HARA-functions with $\gamma > \phi$ as a very good approximation if asset returns preclude approximate arbitrage as defined by Bernado / Ledoit (2000). The intuition for this result can be obtained from the two fund-separation property of the HARA-class. The optimal portfolio is determined by the structure of the risky fund and the volume invested in the risky fund. Separation means that, given the exponent γ of the utility function, the structure of the optimal risky fund is not affected by the investor's initial endowment and the constant in the utility function. A change in γ changes the structure. It turns out, however, that this structure change appears to have only a small effect on the certainty equivalent of the optimal portfolio payoff. Therefore the dependence of the structure on γ matters little. It is well known that the optimal volume invested in the risky fund declines whenever γ increases. We demonstrate that the optimal volume declines approximately in a hyperbolic manner in γ. This provides a simple mechanical rule to approximate the optimal volume.

The approximated portfolio turns out to be particularly simple for an investor trading stocks. Suppose that the pricing kernel of the market portfolio has constant elasticity, as implicitly assumed in the Black-Scholes model. Then as is well known from Merton's work (1971), an investor whose constant RRA equals the constant elasticity of the pricing kernel, is fully invested in the market portfolio. We use the market portfolio as an approximation for the risky fund also for HARA-investors with levels of RRA close to or above the constant elasticity of the pricing kernel. This yields a very good approximation.

A similar approach can be used if the elasticity of the pricing kernel is not constant. Various papers estimated the elasticity of the pricing kernel for the market portfolio. Ait-Sahalia / Lo (2000), Jackwerth (2000), Rosenberg / Engle (2002), Bliss / Panigirtzoglou (2004) estimate the elasticity of the pricing kernel using prices of options on the S&P 500 and the FTSE 100. They conclude that the pricing kernel elasticity is declining, perhaps with a local maximum in between. If the pricing kernel of the market portfolio does not have constant elasticity, we derive a transformed market portfolio such that its pricing kernel has constant elasticity. Then, instead of the market portfolio, we use this transformed market portfolio. It can be approximated by the market portfolio and options on the market portfolio with different strike prices or by a dynamic trading strategy.

The approximation cannot be used when the investor's γ is much less than the RRA ϕ used for the approximation. This is not surprising because this investor would invest more than her initial endowment in the (transformed) market portfolio and borrow. Thus, she might end up with negative terminal wealth which is infeasible given constant RRA. Also the approximation is bad when the risky asset returns are strongly positively or negatively correlated. This paves the way for approximate arbitrage opportunities, i.e. the possibility to achieve positive portfolio excess returns with very high probability. Then the ratio of expected portfolio gains over expected portfolio losses is very high. Bernado / Ledoit (2000) use this ratio to characterize approximate arbitrage opportunities. Given such an opportunity, small changes in the utility function matter.

There is an extensive literature about portfolio choice. Hakansson (1970) derives the optimal portfolio for a HARA-investor in a complete market. Regarding dynamic strategies, Merton (1971) was one of the first to look into these strategies in a continuous time model. Later on, Karatzas et. al. (1986) provide a rigorous mathematical treatment of these strategies. They pay attention, in particular, to non-negativity constraints for consumption. Viceira (2001) discusses dynamic strategies in the presence of uncertain labor income. He uses an approximation approach to derive a simplified strategy which, however, deviates very little in terms of the certainty equivalent from the optimal strategy. Other papers, for example, Balduzzi / Lynch (1999), Brandt et. al. (2005), look for optimal strategies in the case of asset return predictability, Chacko / Viceira (2005) analyze the impact of stochastic volatility in incomplete markets. Brandt et. al. (2009) derive optimal portfolios using stock characteristics like the firm's capitalization and book-to-market ratio.

Another strand of literature investigates the sensitivity of optimal portfolios to asset return parameters. Black / Littermann (1992) show that the optimal portfolio for a (μ, σ)-investor reacts very sensitively to changes in asset return parameters. Yet, the Sharpe-ratio may vary only little. Then an intensive discussion on shrinkage-models started. Recently, DeMiguel / Garlappi / Uppal (2009) compare several portfolio strategies to the simple $1/n$ strategy that gives equal weight to all risky investments. These strategies are compared out-of-sample in the presence of empirically observed uncertainties about asset return parameters. Using the certainty equivalent return for an investor with a quadratic utility function, the Sharpe-ratio and the turnover volume of each strategy as criteria, they find that no strategy consistently outperforms the $1/n$ strategy. In a related paper, DeMiguel / Garlappi / Nogales / Uppal (2009) solve for minimum-variance-portfolios under additional constraints. They find that a partial minimum-variance portfolio calibrated by maximizing the portfolio return in the previous period performs best out-of-sample. Jacobs / Müller / Weber (2009) compare various asset allocation strategies including stocks, bonds and commodities and find that a broad class of asset allocation strategies with fixed weights for the asset classes performs out-of-sample equally well in terms of the Sharpe-ratio as long as strong diversification is maintained. Including transaction costs, strategies with strong turnover perform worse. Hodder / Jackwerth / Kolokolova (2009) find that portfolios based on second order stochastic dominance perform best out-of-sample. Summarizing, these papers find that strong diversification and low turnover matter in the presence of uncertainty about asset return parameters. The papers disagree on whether simple allocation strategies are as good as sophisticated strategies.

The rest of the paper is organized as follows. Section 2 and 3 describe the general approximation approach and the measurement of the approximation quality.

Section 4 shows how the approximation approach works in a perfect market with a continuous state space and long investment horizons. In section 5, we consider a market with a discrete state space to better understand the limitations of the approximation approach. In particular, we consider portfolios with few correlated loans. Section 6 concludes.

2 The Approximation Approach

The approximation derives the optimal portfolio of a HARA-investor with declining absolute risk aversion from the portfolio which would be optimal for an investor with low constant RRA. We consider a market with n risky assets and with one risk-free asset. The gross return of asset i is denoted R_i for $i \in \{1, \ldots, n\}$. We denote the vector $(R_1, \ldots, R_n)'$ by $\mathbf{R}$. The gross risk-free rate is R_f. An investor with initial endowment W_0 maximizes her expected utility of payoff V, given by

$$V := V(\alpha, W_0) = (W_0 - \alpha'\mathbf{1})R_f + \alpha\mathbf{R} = W_0R_f + \alpha\mathbf{r},$$

where α_i denotes the dollar-amount invested in asset i, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\mathbf{1}$ is the n-dimensional vector consisting only of ones. $r_i = R_i - R_f$ denotes the random excess return of asset i and $\mathbf{r} = (r_1, \ldots, r_n)'$. We assume that the investor has a utility function with hyperbolic absolute risk aversion

$$u(V) = \frac{\gamma}{1-\gamma}\left(\frac{\eta + V}{\gamma}\right)^{1-\gamma}, \tag{1}$$

where η and $\gamma < \infty$ are such that u is increasing and concave in V. Moreover, $\gamma > 0$ indicates decreasing absolute risk aversion. For $\gamma = 1$, we obtain log-utility. The first order condition for this optimization problem is

$$\mathbb{E}\left[r_i\left(\frac{\eta + W_0R_f + \alpha^+\mathbf{r}}{\gamma}\right)^{-\gamma}\right] = 0, \ \forall i \in \{1, \ldots, n\}. \tag{2}$$

The optimal solution is denoted α^+.

Our approximation approach consists of the following three steps. First, we transform the decision problem to an equivalent problem under constant RRA. Define $\tilde{W}_0 = \frac{\eta}{R_f} + W_0$ as the enlarged initial endowment. Then after substituting $\tilde{W}_0$ in (2), this condition remains the same, but the investor is constant relative risk averse. Second, we restrict the enlarged initial endowment to the artificial level $\frac{\gamma}{R_f}$. This leaves the structure of the optimal portfolio unchanged. Without loss of generality, we multiply the first order condition (2) by $(\tilde{W}_0R_f/\gamma)^{\gamma}$. This gives

$$\mathbb{E}\left[r_i\left(1 + \frac{\hat{\alpha}^+\mathbf{r}}{\gamma}\right)^{-\gamma}\right] = 0, \ \forall i \in \{1, \ldots, n\} \tag{3}$$

where $\hat{\alpha}^+ = \alpha^+ \frac{\gamma}{\tilde{W}_0 R_f} = \alpha^+ \frac{\gamma}{\eta + W_0 R_f}$. Equation (3) represents the first order condition for an investor with constant relative risk aversion γ and the artificial initial endowment $\frac{\gamma}{R_f}$. $\hat{V}^+$ is the optimal terminal wealth for initial endowment γ/R_f, V^+ for the enlarged initial endowment $\tilde{W}_0$. The terminal wealth implied by (3) is

$$\hat{V}^+ = V\left(\hat{\alpha}^+, \frac{\gamma}{R_f}\right) = \gamma + \hat{\alpha}^+ \mathbf{r} > 0. \tag{4}$$

Positivity follows from $u'(\hat{V}^+) \to \infty$ for $\hat{V}^+ \to 0$.

The solution of the optimization problem for an investor with enlarged initial endowment $\tilde{W}_0$ is proportional to that with artificial endowment $\frac{\gamma}{R_f}$: $V^+ = \hat{V}^+ \tilde{W}_0 R_f / \gamma$. The structure of the risky portfolio is not affected.

Third, we define some low level of constant relative aversion ϕ to approximate the structure of the optimal portfolio. We approximate the solution of equation (3), $\hat{\alpha}^+$, by $\hat{\alpha}^-$, the solution of

$$\mathbb{E}\left[r_i \left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\phi}\right)^{-\phi}\right] = 0, \ \forall i \in \{1, \ldots, n\}. \tag{5}$$

The terminal wealth implied by (5) is $\phi + \hat{\alpha}^- \mathbf{r} > 0$. Comparing $\hat{\alpha}^+$ and $\hat{\alpha}^-$ reveals two effects, a structure effect and a volume effect. The structure of α is defined by $\alpha_1 : \alpha_2 : \alpha_3 : \ldots : \alpha_n$. This structure changes with the level of RRA used for optimization. This structure change is denoted the structure effect. The volume is defined as the amount of money invested in all risky assets together. Hence the volume equals $\sum_{i=1}^{n} \alpha_i$. This volume also changes when RRA ϕ replaces RRA γ. The volume change is denoted the volume effect. Note that the volume effect is not only driven by changing the level of RRA, but also by changing the artificial endowment from ϕ/R_f to γ/R_f. Hence, to make up for the difference in initial endowment in our approximation, the difference, $\gamma/R_f - \phi/R_f$, is simply invested in the risk-free asset adding $\gamma - \phi$ to the terminal wealth $\phi + \hat{\alpha}^- \mathbf{r}$

$$\hat{V}^- = \phi + \hat{\alpha}^- \mathbf{r} + (\gamma - \phi) = \gamma + \hat{\alpha}^- \mathbf{r} \tag{6}$$

Note that $\phi > \gamma$ can lead to negative terminal wealth for low values of $\phi + \hat{\alpha}\mathbf{r}$. Then utility is no longer defined. Since the investor is less risk averse than the investor with RRA ϕ, she should not choose a less risky portfolio. Therefore, we propose that she puts all her money into the portfolio which is optimal for RRA ϕ. This assures positive terminal wealth

$$\hat{V}^- = \frac{\gamma}{\theta}\left(\theta + \hat{\alpha}^- \mathbf{r}\right); \ \gamma < \phi, \tag{7}$$

while we use the approximation (6) for $\gamma \geq \phi$. The paper focusses on the case $\gamma \geq \phi$, but will also present results for $\gamma < \phi$, using equation (7). Rescaling the terminal wealth to the enlarged initial endowment is the same for $\hat{V}^+$ and $\hat{V}^-$. $\hat{V}^+$ and $\hat{V}^-$ are premultiplied by $\tilde{W}_0 R_f / \gamma$.

3 The Approximation Quality

3.1 The General Argument

First, we present some arguments which support our conjecture of a strong approximation quality. Comparing (4) and (6) shows $\hat{V}^+ - \hat{V}^- = (\hat{\alpha}^+ - \hat{\alpha}^-)\,\mathbf{r}$. Hence we can expect a good approximation of the optimal terminal wealth $\hat{V}^+$ if the vectors $\hat{\alpha}^+$ and $\hat{\alpha}^-$ are similar. In other words, a good approximation is obtained if the structure effect and the volume effect are small. Obviously, both effects disappear for $\gamma = \phi$. But we would like both effects to be rather small even if γ is much higher than ϕ. Essential for this is that both utility functions display similar patterns of absolute risk aversion in the range of relevant terminal wealth. Comparing the utility functions

$$\frac{\gamma}{1-\gamma}\left(\frac{\gamma+\alpha\mathbf{r}}{\gamma}\right)^{1-\gamma} \text{ and } \frac{\phi}{1-\phi}\left(\frac{\phi+\alpha\mathbf{r}}{\phi}\right)^{1-\phi}$$

gives absolute risk aversion functions

$$\frac{1}{1+\alpha\mathbf{r}/\gamma} \text{ and } \frac{1}{1+\alpha\mathbf{r}/\phi}.$$

$\alpha\mathbf{r}/\gamma$, respectively $\alpha\mathbf{r}/\phi$, denotes the portfolio excess return, divided by R_f. Hence, if the portfolio excess return is zero, both utility functions display absolute risk aversion of 1. As long as the portfolio excess return does not differ much from 0, absolute risk aversion is similar for both functions implying similar portfolio choice. This follows because risk aversion is declining for all HARA-functions considered. Figure 1 illustrates the absolute risk aversion functions for different levels of γ. The smaller is γ, the steeper the curve is. For exponential utility, the curve is horizontal at a level of 1. The similar patterns for absolute risk aversion suggest a small structure effect.

Why do we expect a small volume effect for this approach? The optimal portfolio $\hat{\alpha}^+$, derived from (3), is well approximated by $\hat{\alpha}^-$ for a large range of γ because an increase in γ starting at ϕ has two opposing sub-effects which balance each other to a large extent. First, due to the impact of risk aversion on portfolio choice, the amount invested in the risky portfolio, given the endowment, decreases roughly hyperbolically with increasing γ for $\gamma \geq \phi$. Second, the artificial initial endowment $\frac{\gamma}{R_f}$ increases with γ, which translates into a proportional increase in the portfolio $\hat{\alpha}^+$. Hence, both sub-effect neutralize each other to a large extent so that the volume effect should be small.

The first order conditions (3) and (5) allow us to derive more precisely market settings of high approximation quality. Let $u^i(\cdot)$ denote the i-th derivative of the utility function. Then a Taylor series for the first derivative of the utility function

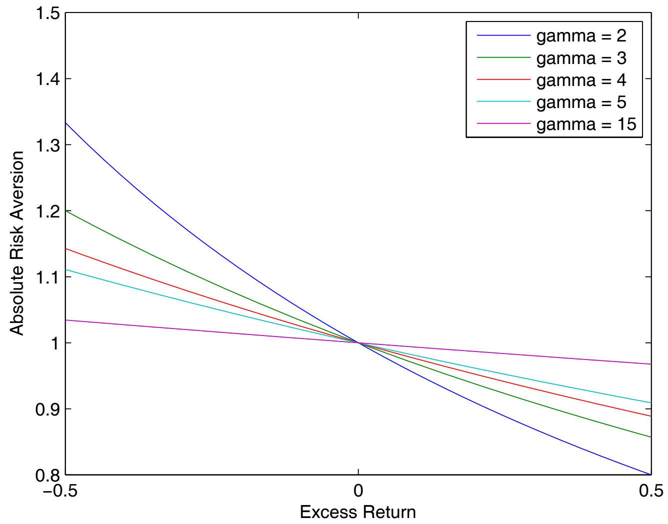

Figure 1: The absolute risk aversion of the HARA-function with endowment γ/R_f declines in the portfolio excess return. For increasing γ the difference between the absolute risk aversion of the HARA-function and that of the exponential utility function, being 1 everywhere, decreases.

around an excess return of zero yields

$$u'(\hat{\alpha}^{+}\mathbf{r}) = \sum_{i=0}^{\infty} \frac{u^{(i+1)}(0)}{i!} (\hat{\alpha}^{+}\mathbf{r})^{i} \tag{8}$$

so that

$$\left(1 + \frac{\hat{\alpha}^{+}\mathbf{r}}{\gamma}\right)^{-\gamma} = 1 + \sum_{i=1}^{\infty} (-1)^{i} \frac{(\hat{\alpha}^{+}\mathbf{r})^{i}}{i!} \prod_{j=0}^{i-1} \left(\frac{j}{\gamma} + 1\right). \tag{9}$$

Hence, the first order condition (3) can be rewritten as

$$\begin{aligned} \mathbb{E}\left[\hat{\alpha}^{+}\mathbf{r}\left(1 + \sum_{i=1}^{\infty} (-1)^{i} \frac{(\hat{\alpha}^{+}\mathbf{r})^{i}}{i!} \prod_{j=0}^{i-1} \left(\frac{j}{\gamma} + 1\right)\right)\right] &= 0 \\ \Leftrightarrow \quad \mathbb{E}[\hat{\alpha}^{+}\mathbf{r}] + \sum_{i=1}^{\infty} (-1)^{i} \frac{\mathbb{E}[(\hat{\alpha}^{+}\mathbf{r})^{i+1}]}{i!} \prod_{j=0}^{i-1} \left(\frac{j}{\gamma} + 1\right) &= 0. \end{aligned}$$

Denoting the i-th non-centered moment of the optimal portfolio excess return by m_i and rearranging the last equation, the first order condition (3) can be rewritten as

$$\frac{m_1}{m_2} + \frac{1}{2}\frac{m_3}{m_2}\left(\frac{1}{\gamma} + 1\right) - \frac{1}{6}\frac{m_4}{m_2}\left(\frac{2}{\gamma} + 1\right)\left(\frac{1}{\gamma} + 1\right) + \ldots = 1. \tag{10}$$

The first order condition (5) yields

$$\frac{n_1}{n_2} + \frac{1}{2}\frac{n_3}{n_2}\left(\frac{1}{\phi} + 1\right) - \frac{1}{6}\frac{n_4}{n_2}\left(\frac{2}{\phi} + 1\right)\left(\frac{1}{\phi} + 1\right) + \ldots = 1, \tag{11}$$

where n_i is the i-th non-centered moment of $\hat{\alpha}^-\mathbf{r}$. For given γ, portfolio excess returns below 1 imply $|\alpha\mathbf{r}|^{i+1} < |\alpha\mathbf{r}|^i$. Hence, it follows for the non-centered moments: $|m_{i+2}| << |m_i|,\ i \geq 2$. Also, $|m_3| << m_2$. Therefore, we may neglect the terms $m_i,\ i \geq 5$ in the Taylor series and focus on the first four moments. Whenever the excess return distribution of the optimal portfolio has non-centered third and forth moments close to zero, both first order conditions are very similar implying a very good approximation quality[4]. Market settings for which the optimal portfolio has a fat tailed and wide distribution may induce a low approximation quality.

By comparing the first order conditions (10) and (11), it can be seen that the approximated return distribution derived from (11) attaches too much weight to the skewness and the kurtosis relative to (10) for $\gamma > \phi$. Hence, we expect the approximated return distribution to have fatter tails, but less skewness than the optimal return distribution. This follows because a HARA-investor with declining absolute risk aversion likes positive skewness, but dislikes kurtosis.

We summarize our findings in the following lemma:

Lemma 1 *The approximation is of high quality even for large differences between ϕ and γ if the non-centered moments of the optimal portfolio excess return decline fast such that $m_{i+2} << m_i,\ i \geq 2,\ m_3 << m_2$ and $n_{i+2} << n_i,\ i \geq 2,\ n_3 \leq n_2$.*

3.2 The Approximation Loss

We measure the economic impact of the approximation by the approximation loss. To do that, we compare the certainty equivalent of the optimal portfolio α^+, derived from solving (2), and that of the approximation portfolio α^-. In both cases, the certainty equivalent is derived using the investor's HARA-function (1). For that utility function, given a portfolio α, the certainty equivalent, CE, is defined by

$$\begin{aligned} \left(\frac{\eta + CE}{\gamma}\right)^{1-\gamma} &= \mathbb{E}\left[\frac{(\eta/R_f + W_0)R_f + \alpha\mathbf{r}}{\gamma}\right]^{1-\gamma} \\ &= \left(\tilde{W}_0\frac{R_f}{\gamma}\right)^{1-\gamma}\mathbb{E}\left[1 + \frac{\hat{\alpha}\mathbf{r}}{\gamma}\right]^{1-\gamma} = \left(\frac{ce}{\gamma}\right)^{1-\gamma}. \end{aligned} \tag{12}$$

Expected utility is the same for an investor with utility function (1) and endowment W_0 and an investor with constant relative risk aversion and enlarged initial endowment $\tilde{W}_0 = \eta/R_f + W_0$. Therefore, we consider the enlarged certainty equivalent $ce = \eta + CE$. Define ε as the ratio of the enlarged certainty equivalent, ce^+, of the optimal portfolio $\alpha^+ = \hat{\alpha}\tilde{W}_0 R_f/\gamma$, and the enlarged certainty equivalent, ce^-, of the

[4]For small portfolio risk, $m_i \to 0$ for $i > 2$. Then the optimal portfolio satisfies $m_1/m_2 \to 1$ rendering γ irrelevant.

approximated optimal portfolio $\alpha^- = \hat{\alpha}^- \tilde{W}_0 R_f/\gamma$. Then

$$\varepsilon = \frac{ce^+}{ce^-} = \left(\frac{\mathbb{E}\left[\left(\frac{(\eta/R_f + W_0)R_f + \alpha^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]}{\mathbb{E}\left[\left(\frac{(\eta/R_f + W_0)R_f + \alpha^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]} \right)^{1/(1-\gamma)} = \left(\frac{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]}{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]} \right)^{1/(1-\gamma)}. \tag{13}$$

Hence, ε is the same for the enlarged initial endowment $\eta/R_f + W_0$ and the artificial initial endowment γ/R_f. This is stated in:

Lemma 2 *For a given market setting, the certainty equivalent ratio ε depends on the exponent γ, but not on the initial endowment nor on the parameter η.*

The lower boundary of ε is one, since the optimal portfolio $\hat{\alpha}^+$ yields the highest possible certainty equivalent. For a HARA-investor there exists a second interpretation of ε. $k = (\varepsilon - 1) \geq 0$ is the relative increase in the enlarged initial endowment $\tilde{W}_0$, that is required for the approximated portfolio to generate the same expected utility as the optimal portfolio generates with initial endowment $\tilde{W}_0$. To see that $\varepsilon = 1 + k$, note

$$\left(\frac{\tilde{W}_0 R_f}{\gamma} \right)^{\gamma-1} \mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right] = \left(\frac{(1+k)\tilde{W}_0 R_f}{\gamma} \right)^{\gamma-1} \mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right].$$

Rearranging yields

$$1 + k = \left(\frac{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]}{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]} \right)^{1/(1-\gamma)} = \frac{ce^+}{ce^-} = \varepsilon.$$

We call k the approximation loss. If $k = 0.02$, for example, then the investor needs to invest additionally 2% of his enlarged initial endowment in the approximated portfolio to achieve the same expected utility as her optimal portfolio does.

For $\gamma = \phi$, the approximation loss is 0, by definition. If we increase γ, the approximation loss will be positive. But it does not increase monotonically. Instead, for $\gamma \to \infty$, $k \to 0$ again. The exponential utility investor buys a risky portfolio independently of her initial endowment. Given an infinite artificial initial endowment, this risky portfolio does not matter for the certainty equivalent. The same is true for the approximated wealth $\hat{V}^-$. Hence, both certainty equivalents converge for $\gamma \to \infty$ so that $k \to 0$.

In the following, we illustrate the approximation loss k by looking, first, at a complete market with a continuous state space and different distributions of the logarithmic market return. Thereafter, we consider a discrete state space. Since ε is the same for the enlarged initial endowment $\tilde{W}_0$ and the artificial endowment γ/R_f, we always use the latter.

4 Approximation in a Continuous State Space

4.1 Demand Functions for State-Contingent Claims

4.1.1 Characterization of Demand Functions

We start from a perfect market with a continuous state space. First, we consider a complete market. Second, we look at an incomplete market. In a complete market state-contingent claims for all possible states $s \in \mathcal{S}$ exist. Hakansson (1970) was the first to investigate investment and consumption strategies of HARA investors in a complete market. Assume an investor with constant relative risk aversion γ and artificial initial endowment γ/R_f. The investor's demand for state-contingent claims, $\hat{\alpha} = (\hat{\alpha}_s)_{s\in\mathcal{S}}$, is optimized

$$\max_{\hat{\alpha}} \mathbb{E}\left[u(V(\hat{\alpha}, \tilde{W}_0))\right] = \max_{\hat{\alpha}} \mathbb{E}\left[\frac{\gamma}{1-\gamma}\left(\frac{\gamma+\hat{\alpha}}{\gamma}\right)^{1-\gamma}\right] \; s.t. \; \mathbb{E}[\pi V] = \gamma/R_f, \qquad (14)$$

where $\pi = (\pi_s)_{s\in\mathcal{S}}$ denotes the pricing kernel and $\hat{\alpha}_s$ is the demand for claims with payoff one in state s and zero otherwise. Differentiating the corresponding Lagrangian with respect to α_s gives the well-known optimality condition for each state

$$u'(\hat{V}(\hat{\alpha}_s, \gamma/R_f)) = \left(\frac{\gamma+\hat{\alpha}_s}{\gamma}\right)^{-\gamma} = \lambda\pi_s, \; s \in \mathcal{S}. \qquad (15)$$

First, we assume that the pricing kernel is a power function of the payoff of the market portfolio. Consequently, the pricing kernel is

$$\pi_s = \frac{1}{R_f}\frac{R_{M,s}^{-\theta}}{\mathbb{E}[R_M^{-\theta}]}, \qquad (16)$$

where $R_{M,s}$ denotes the gross market return in state s and θ is the constant relative risk aversion of the market, i.e. the constant elasticity of the pricing kernel. Hence, we assume a pricing kernel as implied by the Black-Scholes model.

Replacing π_s by (16) and solving (15) for $\hat{V}_s^+ = \gamma + \alpha_s$ yields for finite γ

$$\ln V_s^+ = \frac{\theta}{\gamma}\ln R_{M,s} + a(\gamma) \;\Leftrightarrow V_s^+ = R_{M,s}^{\theta/\gamma}\exp\{a(\gamma)\}. \qquad (17)$$

$a(\gamma)$ depends on the investor's relative risk aversion and is implicitly given by the budget constraint: $\mathbb{E}[R_M^{\theta/\gamma}\exp\{a(\gamma)\}\pi] = \frac{\gamma}{R_f}$. We have

$$\exp\{a(\gamma)\} = \gamma\frac{\mathbb{E}[R_M^{-\theta}]}{\mathbb{E}\left[R_M^{-\theta+\theta/\gamma}\right]} = \frac{\gamma}{\mathbb{E}^Q\left[R_M^{\theta/\gamma}\right]}, \qquad (18)$$

with $\mathbb{E}^Q[\cdot]$ being the expectation operator under the risk neutral probability measure using the pricing kernel $\pi(R_M)$.

The optimal terminal wealth, $\hat{V}^+$, is approximated by $\hat{V}^-$. For $\gamma \geq \phi$, $\hat{V}^-$ is the optimal terminal wealth of some investor with CRRA ϕ and artificial endowment ϕ/R_f, supplemented by the risk-free payoff $(\gamma - \phi)$,

$$\hat{V}_s^- = R_{M,s}^{\theta/\phi} \exp\{a(\phi)\} + (\gamma - \phi). \tag{19}$$

Consider the special case $\phi = \theta$. This implies that $\hat{V}^-$ is linear in R_M and, hence, $\exp\{a(\theta)\} = \theta/\mathbb{E}^Q[R_M] = \theta/R_f$. Then (19) yields

$$\hat{V}_s^- = \frac{\theta}{R_f} R_{M,s} + \gamma - \theta,\ \gamma \geq \theta. \tag{20}$$

The implied portfolio policy is very simple. The investor invests θ/R_f in the market portfolio and $(\gamma - \theta)/R_f$ in the risk-free asset. Since her initial endowment is γ/R_f, she invests the fraction θ/γ of her endowment in the market portfolio and the fraction $1 - \theta/\gamma$ in the risk-free asset.

For $\gamma < \theta$, the policy is even simpler. Invest everything in the market portfolio so that

$$\hat{V}_s^- = \frac{\gamma}{R_f} R_{M,s},\ \gamma < \theta. \tag{21}$$

How does $(\hat{V}_s^+ - \hat{V}_s^-)$ depend on $(\gamma - \phi)$? $\hat{V}^+(R_M) \to \hat{V}^-(R_M)$ for $\gamma \to \phi$. An increase in γ has a structure and a volume effect. First, consider the structure effect. Suppose $\gamma > \phi = \theta$. The functions $\hat{V}^+(R_M)$ and $\hat{V}^-(R_M)$ have two intersections, given a sufficiently large domain of R_M. This follows since both functions have to intersect at least once to rule out arbitrage opportunities. For $R_M \to 0$, $\hat{V}^-(R_M) \to \gamma - \theta$ and $\hat{V}^+ \to 0$. Hence, for $\gamma > \theta$, there exists a $R_M^{(1)}$ such that $\hat{V}^-(R_M) > \hat{V}^+(R_M) \geq 0$ for all $R_M < R_M^{(1)}$. Since $\hat{V}^+(R_M)$ is strictly concave, there also exists a $R_M^{(2)} > R_M^{(1)}$, such that $\hat{V}^-(R_M) > \hat{V}^+(R_M)$ for all $R_M > R_M^{(2)}$. Hence, both functions intersect twice. The demand for state contingent claims is overestimated by the approximation in the very bad states ($[0, R_M^{(1)}]$) and in the very good states ($[R_M^{(2)}, \infty]$) and underestimated in between, as Figure 2, left illustrates. This range-dependent over-/underestimation of the optimal demand characterizes the structure effect.

Now consider $\gamma < \phi = \theta$. Then, from (21) $\hat{V}^- \to 0$ and $\hat{V}^+ \to 0$ for $R_M \to 0$. The strict convexity of $V^+(R_M)$ and the no arbitrage assumption implies one intersection at a positive R_M-level, ignoring the intersection at $R_M = 0$ (see Figure 2, right). Given only one intersection, the structure effect is likely to be stronger for $\gamma < \phi$ than for $\gamma \geq \phi$.

In addition an increase in γ has a volume effect. With a non-linear optimal demand function, the volume of risk taking is not easily defined. The definition we propose is the slope of the demand function $\hat{V}^+(R_M)$ given a gross return $R_M = 1$. The volume effect of a change in γ then is given by

$$\frac{\partial}{\partial \gamma} \left(\frac{\partial \hat{V}^+}{\partial R_M} \right) \Bigg|_{R_M=1}. \tag{22}$$

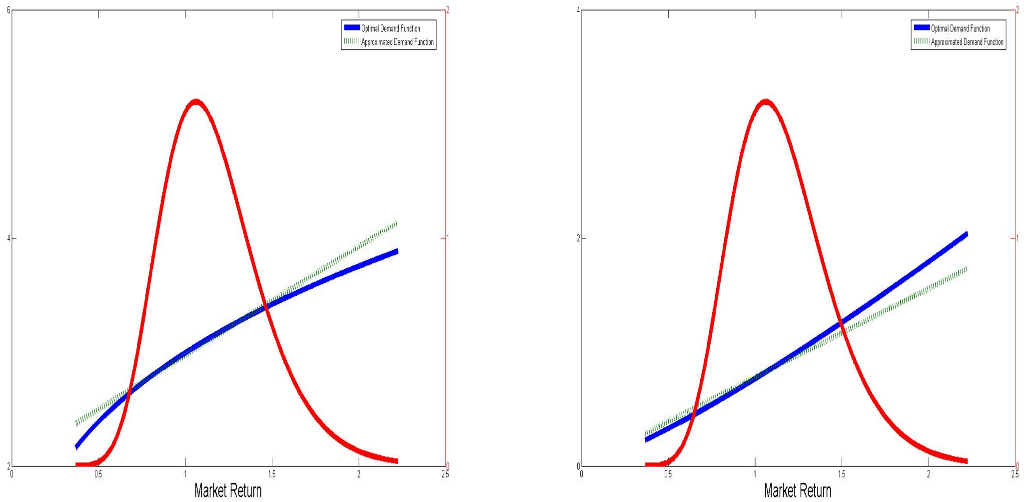

Figure 2: The figure shows the optimal demand for state contingent claims (blue solid curve) and the approximated demand (green dashed line). In addition, the graphs show the probability densities for the market return using the parameters displayed in section 4.2. **Left:** $\gamma \geq \phi$, **Right:** $\gamma < \phi$.

In order to analyze this effect for $\gamma \geq \theta$, we first approximate the optimal demand function by

$$\hat{V}_s^+ \approx R_{M,s}^{\theta/\gamma} \gamma / R_f. \tag{23}$$

Hence we claim for $\gamma \geq \theta$

$$\exp\{a(\gamma)\} = \frac{\gamma}{\mathbb{E}^Q[R_M^{\theta/\gamma}]} \approx \frac{\gamma}{R_f}. \tag{24}$$

For $\theta = \gamma$, $\mathbb{E}^Q[R_M] = R_f$. For $\gamma \to \infty$, $R_M^{\theta/\gamma}$ converges towards a constant so that $\mathbb{E}^Q[R_M^{\theta/\gamma}] \to R_f$. Hence, $\exp\{a(\gamma)\} \to \gamma/R_f$ as well. Simulations indicate that $\exp\{a(\gamma)\}$ is always close to γ/R_f for $\gamma \geq \theta$. With this approximation, (17) yields

$$\begin{aligned}
\frac{\partial \hat{V}^+}{\partial R_M} &\approx \frac{\theta}{\gamma} R_M^{\theta/\gamma - 1} \frac{\gamma}{R_f} \\
&= \frac{\theta}{R_f} R_M^{\phi/\gamma - 1} = \frac{\theta}{R_f} \text{ for } R_M = 1.
\end{aligned}$$

Hence the volume of risk taking is θ/R_f. As this term is independent of γ, γ-changes produce almost no volume effect. The intuition for this result is explained by the two effects which neutralize each other to a large extent: First, an increase in γ, interpreted as the investor's relative risk aversion, makes the investor more cautious, decreasing risk taking approximately proportional to $1/\gamma$. Second, the increase in the initial endowment induces the investor to invest proportionally more in each asset.

Notice, that for $\gamma \geq \phi = \theta$, the slope $\hat{V}^-(R_M)$ also equals θ/R_f. Therefore, the main difference between $\hat{V}^+(R_M)$ and $\hat{V}^-(R_M)$ is driven by the structure effect. This is not true for $\gamma < \theta$ as follows from equation (21). Here, the slope $\hat{V}^-(R_M)$ equals γ/R_f implying a strong volume effect. But also $\exp\{a(\gamma)\}$ tends to be clearly smaller than γ/R_f for $\gamma < \theta$ as illustrated by simulations.

4.1.2 Approximation Quality and Probability Distribution

Next, we try to find out how the approximation quality is affected by changing the shape of the market return distribution. A change in the probability distribution of R_M implies an adjustment in the intersection point(s) of $\hat{V}^+(R_M)$ and $\hat{V}^-(R_M)$. This adjustment tends to stabilize the approximation quality. To characterize the adjustment, we state the following Lemma:

Lemma 3 *a)* $\gamma \geq \theta$. *Let p be the changing parameter of the market return distribution and $R := R(p) \in \{R^{(1)}(p); R^{(2)}(p)\}$, where $R^{(1)}(p)$ / $R^{(2)}(p)$ is the lower / upper intersection point of $\hat{V}^+(R_M|p)$ and $\hat{V}^-(R_M|p)$. Then, holding $\mathbb{E}^Q[R_M] = R_f$ constant*

$$\frac{\partial \ln R}{\partial p} = \frac{\partial a(\gamma)}{\partial p} \frac{R_f \gamma \, \hat{V}^+(R)}{\theta(\gamma - \theta)(R - R_f)}. \tag{25}$$

Since $\hat{V}^+$ is always positive and $R^{(1)}(p) - R_f < 0$ and $R^{(2)}(p) - R_f > 0$, a marginal change in the underlying probability distribution function of R_M

1. *either lowers $R^{(1)}(p)$ and raises $R^{(2)}(p)$,*
2. *or raises $R^{(1)}(p)$ and lowers $R^{(2)}(p)$,*
3. *or leaves $R^{(1)}(p)$ and $R^{(2)}(p)$ unchanged.*

b) $\gamma < \theta$. *Then the change in the positive intersection point is given by*

$$\frac{\partial \ln R(p)}{\partial p} = \frac{a(\gamma)}{\partial p} \frac{\gamma}{\gamma - \theta}.$$

This Lemma is proved in the appendix. To study the effect of a change in p, we need to know $\frac{\partial a(\gamma)}{\partial p}$. Let $f^Q(R_M)$ denote the risk-neutral probability density of R_M.

Lemma 4 *Let p be the changing parameter in the market return distribution. Then*

$$\gamma \frac{\partial a(\gamma)}{\partial p} = \int_0^\infty \left[\hat{V}^-(R_M) - \hat{V}^+(R_M)\right] \frac{\partial f^Q(R_M)}{\partial p} dR_M. \tag{26}$$

The proof is given in the appendix. For illustration, consider a mean preserving spread in the market return, such that $\mathbb{E}^Q[R_M] = R_f$ stays the same. Lemma 1 suggests that the approximation loss increases. However, combining Lemma 3 and Lemma 4 for $\gamma \geq \theta$, we see that an increase in the volatility lowers $R^{(1)}$ and increases $R^{(2)}$. $\hat{V}^+$ is larger than $\hat{V}^-$ in the center of the distribution but smaller in the tails (see Figure 2, left). Increasing the volatility reduces the probability mass in the center and allocates it to the tails. Hence, the integral in equation (26) is positive and $\frac{\partial \ln R}{\partial p}$ (equation (25)) is positive for $R^{(2)}$ and negative for $R^{(1)}$. Therefore, the distance between $R^{(1)}$ and $R^{(2)}$ increases. Hence the claim difference $(\hat{V}^+ - \hat{V}^-)$ is reduced in the tails and raised in the center. Since tail events now have a higher probability, the spreading of the intersection points stabilizes the approximation quality. For $\gamma < \theta$, this stabilizing effect need not exist since a mean preserving spread need not change $a(\gamma)$ (see Figure 2, right).

Alternatively, consider a reduction in the skewness of the market return distribution. For $\gamma \geq \theta$, it is not obvious whether the intersection points are spreading. Relocation probability mass from the right to the left tail of the market return distribution would lower $\mathbb{E}^Q[R_M]$ and, therefore, is infeasible. In order to keep $\mathbb{E}^Q[R_M] = R_f$ unchanged, the probability mass needs to go up in some range of R_M with $R_M > R_f$. Therefore, given $\gamma > \theta$, $a(\gamma)$ can change in either direction. For $\gamma < \theta$, $\frac{\partial a(\gamma)}{\partial p}$ is likely to be positive so that the intersection point declines. That would stabilize the approximation quality.

4.1.3 Non-Constant Elasticity of the Pricing Kernel

Assuming constant elasticity of the pricing kernel for the market return appears restrictive. Empirical studies[5] suggest that the elasticity $\nu(R_M) = -\partial \ln \pi(R_M)/\partial \ln R_M$ is mostly positive and declining. Assume $\nu(R_M) > 0$. Then there exists a transformed market portfolio which has constant elasticity $\tilde{\theta}$. We call this transformed market portfolio the benchmark portfolio BM with return R_{BM}. Let $R_{BM} := g(R_M) := \exp\left\{\frac{1}{\tilde{\theta}} \int_\varepsilon^{R_M} \nu(R_M^0) d \ln R_M^0\right\}$. ε is a positive lower bound of R_M. g is invertible and yields a pricing kernel $\tilde{\pi}$ of constant elasticity $\tilde{\theta}$ with respect to R_{BM}. This follows since

$$\begin{aligned} -\ln \pi(R_M) &= \int_\varepsilon^{R_M} \nu(R_M^0) d \ln R_M^0 \\ &= \tilde{\theta} \ln g(R_M) \\ &= \tilde{\theta} \ln R_{BM} \\ &=: -\ln \tilde{\pi}(R_{BM}). \end{aligned}$$

[5] See, for example, Ati-Sahalia / Lo (2000), Jackwerth (2000), Bliss / Panigirtzoglou (2004), Barone-Adesi / Engle / Mancini (2008). Also the smile effect observed in option markets is consistent with declining elasticity.

Hence, if the elasticity of the pricing kernel on R_M is positive and non-constant, there exists a one-to-one transformation of the market return to the benchmark portfolio return so that constant elasticity of the pricing kernel is recovered. This is true regardless of the sign of $\nu'(R_M)$. As a consequence, our approximated portfolio return is linear in the return of the benchmark portfolio. Moreover, the benchmark portfolio can also be used to obtain a lower pricing kernel elasticity $\tilde{\theta}$ if the pricing kernel elasticity of the market portfolio , θ, is constant, but rather high. Then the benchmark portfolio return is given by $R_{BM} = cR_M^{\theta/\tilde{\theta}}$, with $c > 0$. This benchmark portfolio can be chosen such that for $\gamma \geq \tilde{\theta}$ we obtain a high approximation quality.

4.2 Normal and Other Distributions of Logarithmic Market Returns

4.2.1 Normal Distribution

Now we illustrate the approximation loss numerically assuming that $\ln R_M$ is normally distributed with mean μ and variance σ^2. For a log-normally distributed random variable R_M

$$\mathbb{E}\left[R_M^p\right] = \exp\{p\mu + \frac{1}{2}p^2\sigma^2\}.$$

Hence, we obtain from (17)

$$\begin{aligned} \hat{V}_s^+ &= \gamma R_{M,s}^{\theta/\gamma} \frac{\mathbb{E}\left[R_M^{-\theta}\right]}{\mathbb{E}\left[R_M^{-\theta+\theta/\gamma}\right]} = \gamma \exp\left\{\frac{\theta}{\gamma}\left[\ln R_{M,s} - \mu + \frac{1}{2}\sigma^2\left(2\theta - \frac{\theta}{\gamma}\right)\right]\right\} \\ &= \gamma \left(\frac{R_{M,s}}{R_f}\right)^{\theta/\gamma} \exp\left\{\frac{\sigma^2}{2}\frac{\theta}{\gamma}\left(1 - \frac{\theta}{\gamma}\right)\right\}, \end{aligned} \tag{27}$$

where $R_f = \exp\{r_f\}$. The second line in (27) follows from $\theta = \frac{\mu+\sigma^2/2-r_f}{\sigma^2}$. Note that $\ln \mathbb{E}[R_M] = \mu + \frac{\sigma^2}{2}$ so that the annual Sharpe-ratio is

$$\frac{\mathbb{E}[R_M] - R_f}{\sigma(R_M)} = \left[1 - \exp\left\{r_f - \left(\mu + \frac{\sigma^2}{2}\right)\right\}\right] \left(\exp\{\sigma^2\} - 1\right)^{-1/2}.$$

The certainty equivalent of $\hat{V}^+$ has a closed form representation

$$ce(\hat{V}^+) = \gamma \exp\left\{\frac{1}{2}\frac{\sigma^2\theta^2}{\gamma}\right\}.$$

The approximated portfolio is derived for CRRA ϕ. We can choose ϕ. The easiest choice is $\phi = \theta$. Then, for $\gamma \geq \theta$, $\hat{V}_s^-$ is given by (20). To compute the expected utility for the approximation, we have to rely on numerical integration techniques.

For $\gamma < \theta$, $\hat{V}^-(R_M) = \gamma R_M/R_f$. Then $ce(\hat{V}^-) = \gamma \exp\{\frac{\sigma^2}{2}(2\theta - \gamma)\}$. Hence, the approximation loss is

$$k = \exp\left\{\frac{\sigma^2}{2\gamma}(\theta - \gamma)^2\right\} - 1,\ \gamma < \theta. \tag{28}$$

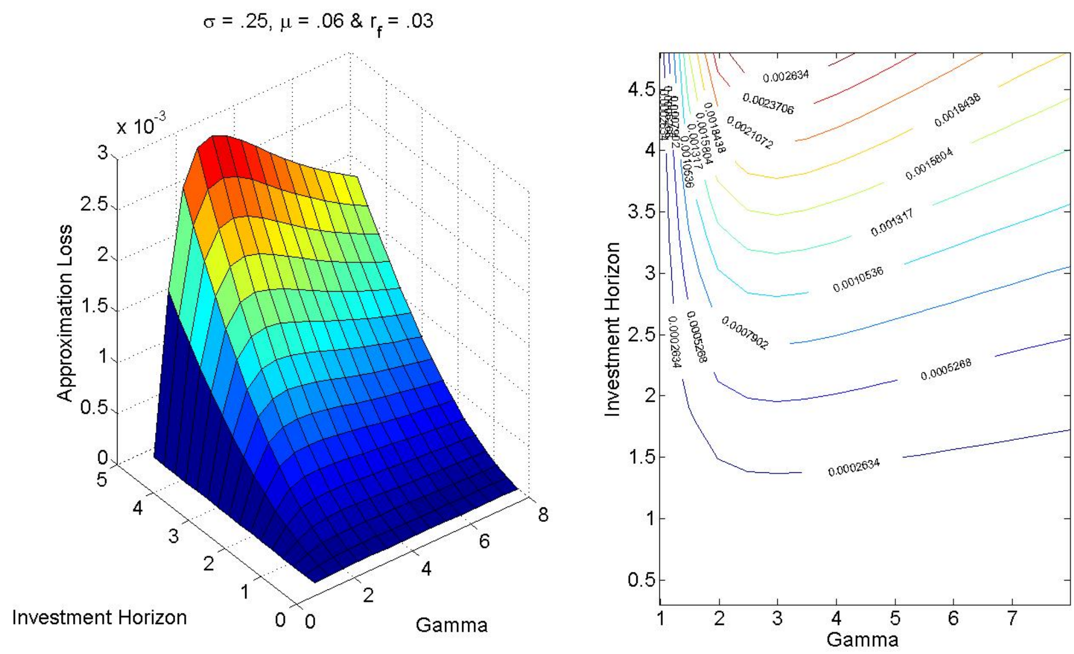

Figure 3: **Left:** The surface shows the approximation loss for $\gamma \in [0.98, 8]$ and for investment horizon between 3 month and five years. For this setting, the highest loss in certainty equivalent is realized for $\gamma \approx 3$ and an investment of five years. The investor would have lost about 0.25% of the optimal certainty equivalent. **Right:** The corresponding isoquants show the combination of γ and investment horizon with the same approximation loss k depicted in each curve.

To calibrate our analysis to observable market returns, we use an annual expected logarithmic market return $\mu = 6\%$ and an annual market volatility $\sigma = 25\%$. The instantaneous risk-free rate is 3%. This implies a pricing kernel elasticity of $\theta = 0.98$, an annual excess return of 6.51% and an annual Sharpe-ratio of 23.4%. We consider investors with constant relative risk aversion in the range $[0.98, 8]$, an investment horizon between three month and 5 years and assume independent increments. Hence, the expected logarithmic market return for t years is $\mu_t = t\mu$ and the standard deviation of the t-year logarithmic market return is $\sigma_t = \sqrt{t}\sigma$.

Figure 3 shows the approximation loss. For $\gamma = 0.98$, the approximated portfolio equals the optimal portfolio so that there is no approximation loss. For $\gamma > \theta$, the approximation loss increases with a longer investment horizon. This follows because for a longer horizon the market return distribution becomes wider implying a higher risk for the investor. The non-centered higher moments of the portfolio excess return relative to the second moment grow implying a lower approximation quality (Lemma 1). Yet, the approximation quality still remains very good. The highest approximation loss in Figure 3, left, is about 0.3% for an investor with γ about 4 and an investment horizon of 5 years. In other words, the investor would need to raise her initial endowment by 0.3% of to make up for the loss of the approximation error. The approximation loss for a given investment horizon has a maximum at some $\gamma > 0.98$ and then monotonically declines with increasing γ to zero. The impact of γ and the investment horizon can also be seen in Figure 3, right, which depicts isoquants of the approximation loss, i.e. combinations of γ and investment horizon yielding the same loss. For an investment horizon of 2.5 years, for example, the loss always remains below 0.07%.

Next, for values of γ below $\phi = 0.98$ the approximation loss is given by equation (28). The approximation loss is relatively high for substantial differences $(\theta - \gamma)$. This is due to the fact that the approximated portfolio always equals the market portfolio without any risk-free lending / borrowing. This inflexibility generates relatively high approximation losses. For example, for $\sigma = 25\%$, $\theta = 0.98$ and $\gamma = \theta/2$, the approximation loss is 1.54%, given an investment horizon of one year. For two years, the loss increases to 3.11%. If, however, $\sigma = 17.5\%$, then the approximation loss is 0.75% for one year and 1.51% for two years.

4.2.2 Symmetric, Fat-tailed Distributions

In the following, we analyze fat-tailed and left skewed distributions and restrict ourselves to $\gamma \geq \theta$. Lemma 1 shows that higher order moments affect the approximation quality substantially. Therefore, we next use the scaled t-distribution to account for excess kurtosis (fat tails) in logarithmic market returns. The density for a t-year investment period is given by

$$f(\ln R_{M,t}|\mu_t, \sigma_{\nu,t}, \nu_t) = \frac{\Gamma\left(\frac{\nu_t+1}{2}\right)}{\sigma_{\nu,t}\sqrt{\nu_t \pi}\,\Gamma\left(\frac{\nu_t}{2}\right)} \left(1 + \frac{\left(\frac{\ln R_{M,t}-\mu_t}{\sigma_{\nu,t}}\right)^2}{\nu_t}\right)^{-(\nu_t+1)/2}, \tag{29}$$

where $\sigma_{\nu,t} = \sigma_t(\nu_t/(\nu_t-2))^{-1/2}$. The mean of the distribution is $\mu_t = t\mu$, the standard deviation is $\sigma_t = \sqrt{t}\sigma$ and the excess kurtosis is $\frac{6}{\nu_t-4}$ for $\nu_t > 4$. Empirical studies, for example Corrado / Su (1997), report a kurtosis of about 12 for the monthly logarithmic returns of the S&P 500 between 1986 and 1995. Assuming independent increments, this translates into an annual kurtosis of 3.75. Independent increments imply that $\kappa_t = \frac{1}{t}\kappa_1 + 3\left(1 - \frac{1}{t}\right)$, where κ_1 is the kurtosis for a one-year time horizon and κ_t is the kurtosis for a t-year time horizon. For robustness, we stress the calculation of the approximation loss with an annual kurtosis of 4.5. This gives the simple rule for ν_t: $\nu_t = 4t + 4$. Using the same parameter values as before, $\mu = 0.06$ and $\sigma = 0.25$, we derive the approximation loss for t-distributed logarithmic market returns, $\gamma \geq 0.98$ and $t \in [0.3; 5]$. By numerical integration, the Sharpe-ratio is 23% for one year and 42% for a five year investment period. The results are shown in Figure 4, left. The fat tails raise the approximation loss, as predicted by Lemma 1. However, the loss in the certainty equivalent is still remarkably low, even for an investment horizon of five years. For $\gamma = 3$ and a five year horizon, the highest approximation loss is less than 0.5%.

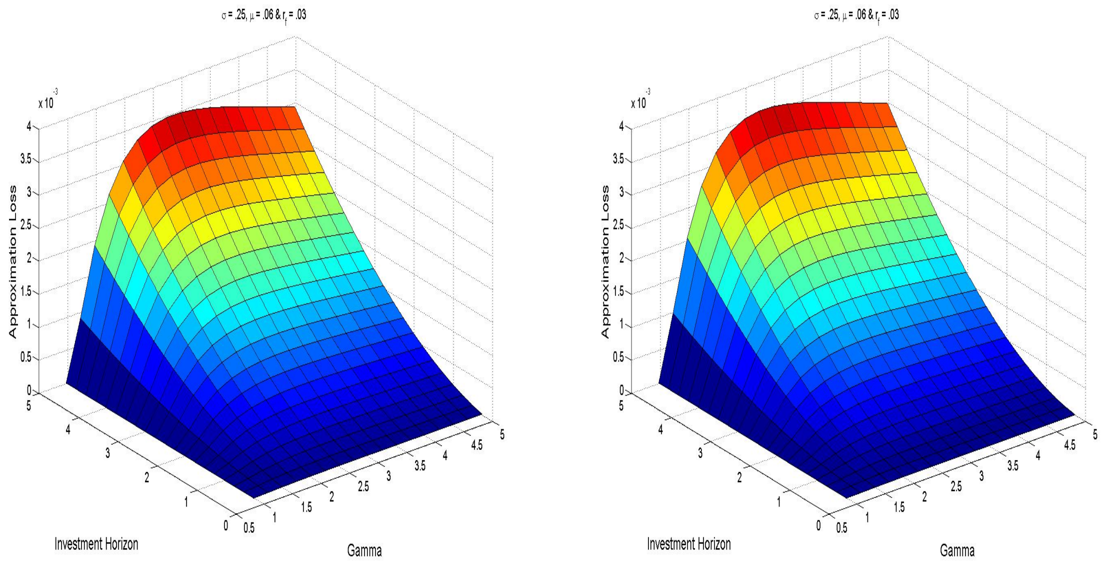

Figure 4: The surface shows the approximation loss as a function of relative risk aversion ($\gamma \geq \theta$) and investment horizon. **Left:** The logarithmic market return is t-distributed. We assume independent and identically distributed increments, hence, $\mu_t = 0.06t, \sigma_t = 0.25\sqrt{t}$ and $\nu_t = 4t + 4$. For $\gamma \approx 3$ and an investment horizon of five years, the highest approximation loss is about 0.4%. **Right:** The logarithmic market return is left-skewed normally distributed with independent and identically distributed increments. Skewness does not affect the approximation loss substantially.

4.2.3 Left-skewed, Fat-tailed Distributions

As a final example of a complete market we consider a distribution with fat tails and negative skewness. Since 1987 stock returns up to one year are mostly skewed to the left. This is also true for stock index returns. Over long periods index returns tend to be skewed to the right. A true probability distribution with negative skewness tends to produce a high Sharpe-ratio because low market returns, associated with high prices for state contingent claims, have high probability. For the simulation we use the skewed normal distribution to model the logarithmic market return. The density function is given by

$$f(\ln R_{M,t}|\lambda_t, \omega_t, \xi_t) = \left(\frac{2}{\sigma_t}\right) n\left(\frac{\ln R_{M,t} - \lambda_t}{\omega_t}\right) \mathcal{N}\left(\xi_t\left(\frac{\ln R_{M,t} - \lambda_t}{\omega_t}\right)\right), \qquad (30)$$

where $n(\cdot)$ is the density of the standard normal density function and $\mathcal{N}(\cdot)$ is the standard normal distribution function. The mean is given by $\mu_t = \lambda_t + \omega_t\delta_t\sqrt{2/\pi}$, the standard deviation is $\sigma_t = \omega_t\sqrt{1 - 2\delta_t^2/\pi}$, where $\delta_t = \xi_t/\sqrt{1+\xi^2}$ [6]. Corrado / Su (1997) find that the monthly logarithmic stock returns of the S&P 500 are skewed to the left by -1.67. Assuming independent increments, this translates to

[6] The skewness is $sk_t = \frac{4-\pi}{2}\frac{\left(\delta_t\sqrt{2/\pi}\right)^3}{(1-2\delta_t^2/\pi)^{3/2}}$ and the excess kurtosis is $2(\pi-3)\frac{\left(\delta_t\sqrt{2/\pi}\right)^4}{(1-2\delta_t^2/\pi)^2}$.

an annual skewness of about -0.5[7]. Again, to check for robustness, we stress this number and use an annual skewness of -0.6. For each investment horizon, we choose the parameters λ_t, ω_t and ξ_t such that $\mu_t = 0.06t, \sigma_t = 0.25\sqrt{t}$, $sk_t = -0.6/\sqrt{t}$ and the excess kurtosis over t years is $(3.4426 - 3)/t$. The approximation loss is shown in Figure 4, right, for $\gamma \geq \theta$ and $t \in [0.3; 5]$. The maximum approximation loss is about 0.4% for $\gamma = 3$ and 5 years. Figure 4, left, and Figure 4, right, indicate similar loss levels. Skewness does not affect the approximation loss substantially. This is driven also by the adjustment of the intersection points of the optimal and the approximated demand functions to the skewness change.

4.3 Generalization to Incomplete Markets

In an incomplete market, the pricing kernel is no longer unique. Suppose, first, that a pricing kernel with low constant elasticity is among the pricing kernels which are feasible in a no-arbitrage market. For this case the preceding analysis has shown that buying the market portfolio and the risk-free asset provides a very good approximation to the optimal portfolio for a large variety of settings. Actually, in an incomplete market the approximation quality is even better. This follows because incompleteness does not affect the availability of the market portfolio and, hence, the approximate policy, but the optimal portfolio in a complete market is no longer available. The optimal portfolio in the incomplete market can be derived from a model for a complete market, subject to additional constraints regarding the availability of claims. These constraints reduce the certainty equivalent of the optimized portfolio, i.e. the numerator of ε while the denominator stays the same. Hence, the approximation loss declines.

Second, suppose that that a pricing kernel with low constant elasticity cannot explain security prices. Then we can use the benchmark portfolio which is a transformed market portfolio. This portfolio as well as the optimal portfolio cannot be replicated exactly by a portfolio of the available assets in an incomplete market. Whether both effects together raise or lower the approximation loss, cannot be answered in general. But, given a large number of available risky assets, the incompleteness effect should be small anyway. This is particularly true in the presence of many options on the market portfolio. Alternatively, dynamic portfolio strategies might be used. The implied asset turnover raises transaction costs for the optimal and the approximation portfolio. Again, the impact on the approximation loss is indeterminate.

[7]Independent increments imply $sk_t = sk_1/\sqrt{t}$, where sk_1 denotes the skewness for one year and sk_t is the skewness for t-years.

5 Approximation in a Discrete State Space

So far, we considered a continuous state space and found a very good approximation quality for $\gamma \geq \theta$. In the following, we analyze the approximation quality in a discrete state space to sharpen our understanding about the limits to the approximation approach.

5.1 Investing in Loans

The strong approximation quality in a continuous state space is driven also by the fact that the probability mass of the optimal and the approximated portfolio payoff is concentrated around the zero excess payoff. Hence, the higher non-centered moments relative to the second moment are fairly small supporting a high approximation quality (Lemma 1). Therefore, we conjecture that the approximation quality is weaker for portfolio returns with more probability mass in the tails. The extreme case is the binomial case, i.e. a case with two states only. This case is most likely to undermine the conditions for strong approximation quality.

As an example, consider a bank which can invest in loans. The bank does not trade stocks. Hence the pricing kernel of the stock market does not matter for the bank. Yet, we assume that the loan market is arbitrage free. Otherwise an optimal loan portfolio does not exist. In a static setting with default risk only, loans either are fully paid or they go into default paying only a recovery amount. Let the recovery amount be non-random so that each loan is characterized by two possible outcomes only. Then probability mass of the loan payoff is *not* concentrated around the zero excess payoff.

First, assume that the bank can invest in many different loans achieving strong portfolio diversification. Then the loss rate of the loan portfolio can be approximated quite well by a lognormal probability distribution. The random loss rate is defined as the random terminal wealth of the portfolio reduced by loan defaults, divided by the terminal wealth in the absence of defaults. Hence, the bank again choses an optimal portfolio such that the probability mass of its excess payoff is concentrated around zero. This suggests again a high approximation quality.

Second, critical are cases in which the number of loans available to the bank is small, say n. Then there exist 2^n different payoffs of the loan portfolio. If n is rather small, then only a few payoffs exist. In this situation, the higher non-centered moments of the portfolio payoff relative to the second moment might become rather high impairing the approximation quality . In the following, we present some examples to demonstrate the structure and the volume effect and the approximation loss. In the first extreme example $n = 1$. Then we consider cases with $n = 2$.

5.1.1 One Risky Asset

In the case of only one risky asset two fund separation is trivial. There is no structure effect. Yet, the volume effect $(\hat{\alpha}^+ - \hat{\alpha}^-)$ remains. We analyze this case to better understand the volume effect. Also it sheds light on the approximation loss. We show that the approximation loss is small if $\gamma > \phi$ and an approximate arbitrage opportunity does not exist.

With one risky asset, α is a scalar. Let $\hat{\alpha}^+ = \hat{\alpha}(\gamma)$ and $\hat{\alpha}^- = \hat{\alpha}(\phi)$ denote the optimal and the approximated amount invested in the risky asset. Lemma 5 shows that the volume effect can be positive or negative:

Lemma 5 *Consider a market with one risky asset and one risk-free asset. Let $\hat{\alpha}^+$ denote the optimal amount invested in the risky asset and $\hat{\alpha}^-$ the approximated amount. Then the sign of the volume effect, $\hat{\alpha}^+ - \hat{\alpha}^-$, is given by*

$$sgn(\hat{\alpha}^+ - \hat{\alpha}^-) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0$$

$$\Leftrightarrow$$

$$cov\left(r, \frac{(1+\hat{\alpha}^- r/\gamma)^{-\gamma}}{\mathbb{E}\left[(1+\hat{\alpha}^- r/\gamma)^{-\gamma}\right]} - \frac{(1+\hat{\alpha}^- r/\phi)^{-\phi}}{\mathbb{E}\left[(1+\hat{\alpha}^- r/\phi)^{-\phi}\right]} \right) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0.$$

This lemma is proved in the appendix. By Lemma 5, the volume effect $(\hat{\alpha}^+ - \hat{\alpha}^-)$ is positive if the covariance between the asset return and the standardized marginal utility for γ is higher than that for ϕ, given the approximate choice $\hat{\alpha}^-$. Equivalently, it is positive if the covariance between the asset return and the difference in standardized marginal utilities is positive. Let $\Delta(\alpha r)$ denote the difference in standardized marginal utilities. Suppose $\gamma > \phi$. Then Δ has the pattern as indicated in Figure 5 and proved in the appendix. It starts at $-\infty$ for $\phi + \hat{\alpha} r \to 0$, then increases with $\hat{\alpha} r$ to a positive level, then declines again to a negative level and approaches 0 from below for $\hat{\alpha} r \to \infty$.

This pattern indicates that the covariance between Δ and $\hat{\alpha}\mathbf{r}$ can be positive or negative. Given a probability distribution for $\mathbf{r}$, skewed to the left (right), we expect a positive (negative) covariance and hence a positive (negative) volume effect. Hence the volume effect is likely to be positive (negative) for a negatively (positively) skewed portfolio return. It should be noted that Lemma 5 approximately holds also for multiple risky assets if the structure effect is small. Then the optimal and the approximated portfolio would have similar structures.

We illustrate Lemma 5 using the case of one risky asset with a binomial return. We consider a negatively and a positively skewed binomial distribution. Both distributions have the same expectation and standard deviation so that the Sharpe-ratio is the same. The risk-free rate is 3%, the annual expected return is 10.5% and the

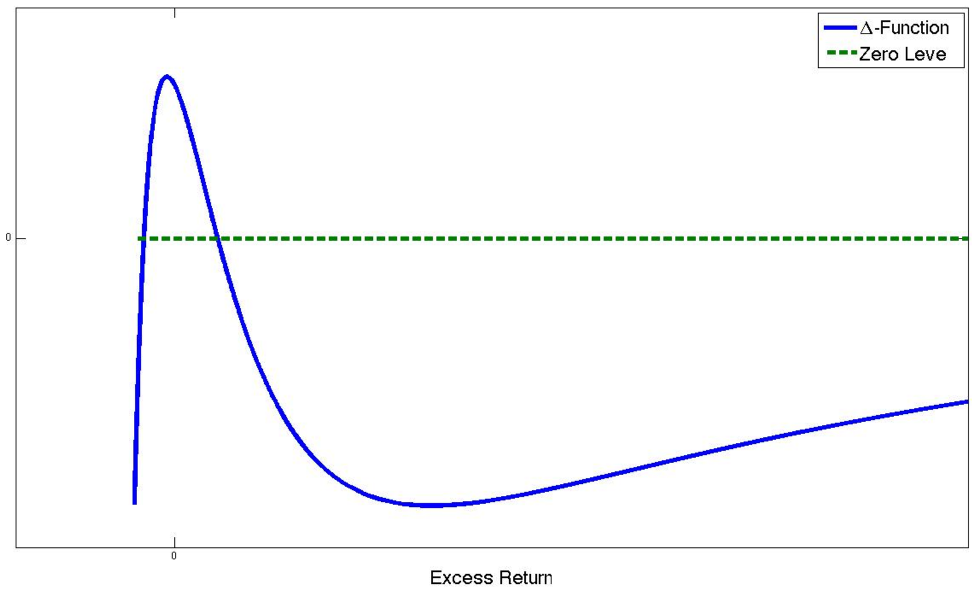

Figure 5: The figure shows the difference in standardized marginal utility, $\Delta(\alpha r)$ for $\gamma > \phi$.

standard deviation is 30%. Let u (d) be the gross return in the up-state (down-state). p is the up-state probability for the distribution R skewed to the right and also the down-state probability for the distribution L skewed to the left. Then we have: $u_R - d_R = u_L - d_L$ and $d_R = d_L + (1 - 2p)(u_R - d_R)$.

For example, let $p = 0.25$, $u_R = 1.42$ and $d_R = 1$. Then $u_L = 1.21$ and $d_L = 0.79$. Hence the distribution R has a skewness of 0.191, while distribution L has a skewness of -0.165. The approximated investment in the risky asset is the optimal investment using $\phi = 1$. The optimal investment, the volume effect and the approximation loss are shown in Table 1 for an investor with constant relative risk aversion $\gamma = 2$, $\gamma = 3$

	$\gamma = 2$		$\gamma = 3$		$\gamma = 10$	
Distribution	R	L	R	L	R	L
$\hat{\alpha}^+$	4.7813	1.8519	4.3083	1.8830	3.7183	1.9187
$(\hat{\alpha}^+ - \hat{\alpha}^-)$	-1.6290	0.1158	-2.1020	0.1469	-2.6920	0.1826
k	0.0038	0.0002	0.0048	0.0002	0.0028	0.0001

Table 1: It shows the optimal investment in the risky asset for $\gamma = 2, 3$ and 10 and the volume effect $(\hat{\alpha}^+ - \hat{\alpha}^-)$. The approximated investment based on $\theta = 1$ is $\hat{\alpha}^- = 6.4103$ for R and $\hat{\alpha}^- = 1.9677$ for L. k is the approximation loss. R (L) denotes the probability distribution skewed to the right (left)

and $\gamma = 10$.

As predicted from Lemma 5, the volume effect is negative (positive) for the positively (negatively) skewed return distribution. The volume effect relative to the optimal investment in the risky asset is rather large (small) for the positively (negatively) skewed distribution. Yet, the approximation loss is rather small in all cases. It is larger for the positively skewed distribution, and it declines with increasing γ for high values of γ.

There is an easy way to understand the strong volume effect for the positively skewed distribution. For a binomial distribution, the first order condition yields

$$p_u r_u \left(1 + \frac{\hat{\alpha}^+ r_u}{\gamma}\right)^{-\gamma} = (1 - p_u)|r_d| \left(1 + \frac{\hat{\alpha}^+ r_d}{\gamma}\right)^{-\gamma}$$

$$\Leftrightarrow \frac{p_u r_u}{(1 - p_u)|r_d|} = \left(\frac{\gamma + \hat{\alpha}^+ r_u}{\gamma + \hat{\alpha}^+ r_d}\right)^{\gamma}. \tag{31}$$

The left hand side of (31) denotes the gain/loss- ratio of Bernado / Ledoit (2000). The higher it is, the closer is an approximate arbitrage opportunity. For the positively (negatively) skewed distribution the gain/loss- ratio is 4.33 (2.25). Hence, the positively skewed distribution is much closer to approximate arbitrage. This explains the stronger volume effect and the weaker approximation quality. Another way to understand this, is to analyze the Arrow-Debreu prices in this complete market setting. For a binomial return there always exists a pricing kernel with constant elasticity. We have

$$\pi_u = \frac{1}{R_f} \frac{p_u R_u^{-\theta}}{\mathbb{E}[R^{-\theta}]} \text{ and } \pi_d = \frac{1}{R_f} \frac{(1 - p_u) R_d^{-\theta}}{\mathbb{E}[R^{-\theta}]}.$$

The ratio $\frac{\pi_u}{\pi_d}$ can be used to solve for θ,

$$\theta = \frac{\ln\Big(\text{gain-loss-ratio}\Big)}{\ln\left(\frac{R_u}{R_d}\right)}.$$

Hence, the pricing kernel elasticity is 4.18 (1.90) for the positively (negatively) skewed distribution. As argued by Bernado / Ledoit, a high elasticity also indicates an approximate arbitrage opportunity.

5.1.2 Two Risky Assets with Dependent Returns

a) Unrestricted Optimization with Binomial Asset Returns

To look into more critical cases regarding the approximation quality, we analyze the case with two risky loans characterized by correlated binomial returns. In this case there exist only 4 states of nature so that the probability mass of a portfolio of these assets is not concentrated around the zero excess return. Therefore the higher non-centered moments relative to the second moment should impair the approximation quality. If there are two loans with different expected returns, and if these are perfectly negatively correlated, then there exists an arbitrage opportunity. If the returns are strongly negatively correlated, then there exists an approximate arbitrage opportunity. Hence, investors will take very large positions in the risky assets which should raise the approximation loss.

For illustration, let the marginal distribution of each risky asset have a binomial distribution with equal probability for both outcomes, the up-state and the down-state. The gross return of asset one is $R_1 = (1.2, 0.925)$ and of asset two is $R_2 = (1.3, 0.85)$, respectively. The risk-free rate is 3%. This implies an expected excess return of 3.25% for asset one and 4.5% for asset two. The standard deviation is 13.75% for the first asset and 22.5% for the second asset. We solve the first order conditions for both assets using the investor's γ for the optimal portfolio and ϕ for the approximation, $\phi \leq \gamma$. Holding the marginal distributions for both asset returns constant, we change the return correlation by the following procedure. Let $P_{s,t} := \mathbb{P}(R_1 = s, R_2 = t)$ denote the probability that asset 1 is in the s-state and asset 2 is in the t-state, $s, t \in \{\text{up, down}\}$. Then, the joint probability assuming a correlation of 1 is

$$[P_{s,t}]_{s,t\in\{\text{up, down}\}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Reducing $P_{\text{up, up}}$ and $P_{\text{down, down}}$ by the same amount and adding this amount to $P_{\text{down, up}}$ and $P_{\text{up, down}}$, decreases the correlation without affecting the marginal distributions.

For relative risk aversions between 0.98 and 8 and for return correlations in [-0.8, 0.8], Figure 6, left, shows the approximation quality. We choose $\phi = 0.98$. Hence, the approximated and the optimal portfolio are the same for $\gamma = 0.98$. The approximation loss is very low for correlations above -0.5, but turns higher for lower correlations. In markets with negatively correlated assets, the investor can buy a hedged portfolio with long positions in both assets and earn a high portfolio return with little downside potential. Consider the case with correlation -0.6 and $\gamma = 2.5$. The optimal portfolio invests about 3.57\$ of the initial endowment in asset 1 and about 2.06\$ in asset 2. This gives an expected excess return of the optimal portfolio of 8.61% and a standard deviation of 17.64%. The approximation invests 3.02\$ in asset 1 and 1.70\$ in asset 2 implying an approximation loss of about 0.15%. The volume effect is $(3.57 + 2.06) - (3.02 + 1.70) = 0.91\$$, it is quite strong in this case. The structure effect $\frac{3.57}{2.06} - \frac{3.02}{1.70} = -0.04$ is, however, very weak. For higher correlations, the approximation quality is excellent.

Figure 7 shows the volume and the structure effect for various constellations of γ and correlation. The volume effect is quite strong for strongly negative correlation, while the structure effect is always quite modest. This indicates that the approximation quality is impaired primarily by the volume effect, not by the structure effect.

As a second example, we assume that the gross return of asset 1 is $R_1 = (1.25, 0.925)$.

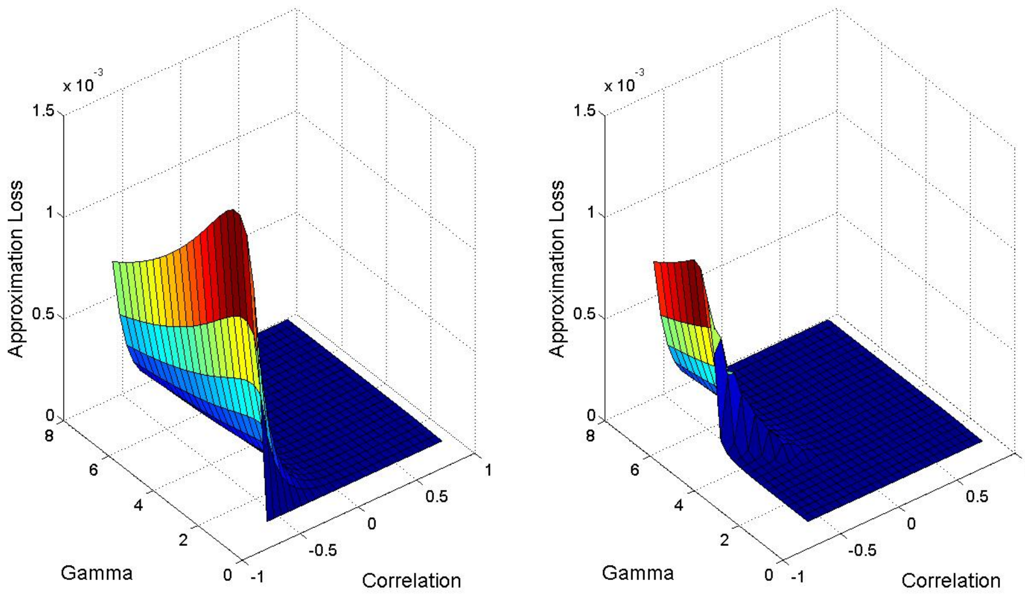

Figure 6: $\gamma > \theta$, correlation between -0.8 and 0.8. **Left:** The approximation loss in a market with two binomial assets is excellent as long as the correlation is above -0.5. The expected excess return for asset 1 is 3.25% and 4.5% for asset 2. The volatility is 13.75% and 22.5%, respectively. **Right:** The figure shows the approximation loss for the same market setting with short selling restrictions. For strongly negatively correlated assets and low relative risk aversion, the restriction becomes binding (for both portfolios) and lowers the approximation loss.

Everything else equal, the expected excess return of asset 1 is 5.75% and the standard deviation is 16.25%. Yet, the expected excess return of asset 1 is higher than the expected excess return of asset 2, but the standard deviation of asset 1 is lower than that of asset 2. Hence, asset 1 taken separately dominates asset 2. Therefore, selling asset 2 to hedge asset 1 is now beneficial when the correlation is strongly positive. The approximation loss for this market setting is shown in Figure 8, left. For strong positive and strong negative correlation the approximation loss increases. This enables the investor to earn high portfolio returns with moderate risk and raises the approximation loss. However, the approximation loss is always below 1% and converges to 0 for high values of γ. Again, the volume effect is quite strong for strongly positive correlation, while the structure effect is always small.

b) Portfolio Restrictions and Binomial Asset Returns

The previous examples showed that the approximation loss becomes substantial whenever the asset correlation provides support for an approximate arbitrage opportunity. Not surprising, in these situations investors take large positions in the risky assets. If they take large positive positions, then they need to borrow a lot. Otherwise, they need to short-sell one risky asset. Short selling and borrowing large

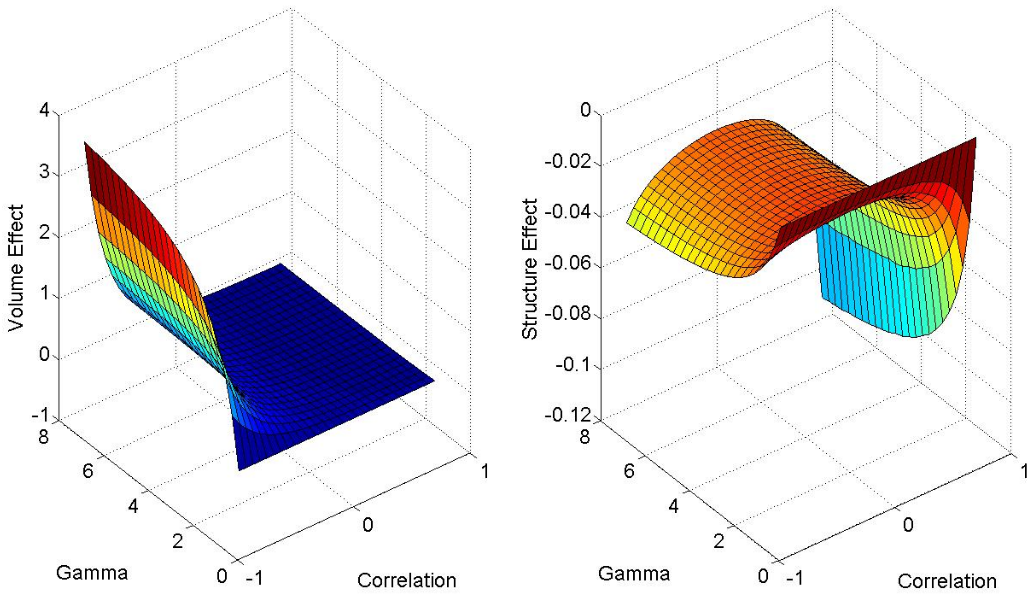

Figure 7: $\gamma > \theta$, correlation between -0.8 and 0.8. **Left:** The volume effect for a market with two binomial assets. Only for strongly negative asset correlation there is a substantial difference between the approximation and the optimal portfolio. **Right:** The structure effect is remarkable small.

amounts is difficult in reality. If we exclude short selling and borrowing, then the optimal and the approximated portfolio are severely constrained. In this situation, we expect the approximation loss to be much smaller. This is illustrated by the following constrained optimization problem

$$\max_{\hat{\alpha}} \mathbb{E}\left[\left(1+\frac{\hat{\alpha}'\mathbf{r}}{\gamma}\right)^{1-\gamma}\right] \; s.t.\; \hat{\alpha} \geq 0,\; \hat{\alpha}'\mathbf{1} \leq \frac{\gamma}{R_f}. \tag{32}$$

The approximation quality improves significantly compared to the market setting without restrictions. This is illustrated in Figure 6, left and right. Both simulations rely on the same market setting, Figure 6, left, gives the approximation loss for an unrestricted market whereas in Figure 6, right, short-selling is prohibited. This surface has a discontinuity that separates the combinations of relative risk aversion and asset return correlation for which the portfolio restrictions are binding and those in which the restrictions are not binding. Compared to Figure 6, left, the restriction decreases the approximation loss strongly in the area $[-0.8, -0.4] \times [1.75, 4]$, where the first dimension is the asset correlation and the second the relative risk aversion. in this area, the restrictions are binding for both, the optimal and the approximate portfolio, so that the approximation loss is small.

c) The $1/n$ Policy

Finally, we compare our approximation to the $1/n$ policy. According to DeMiguel /

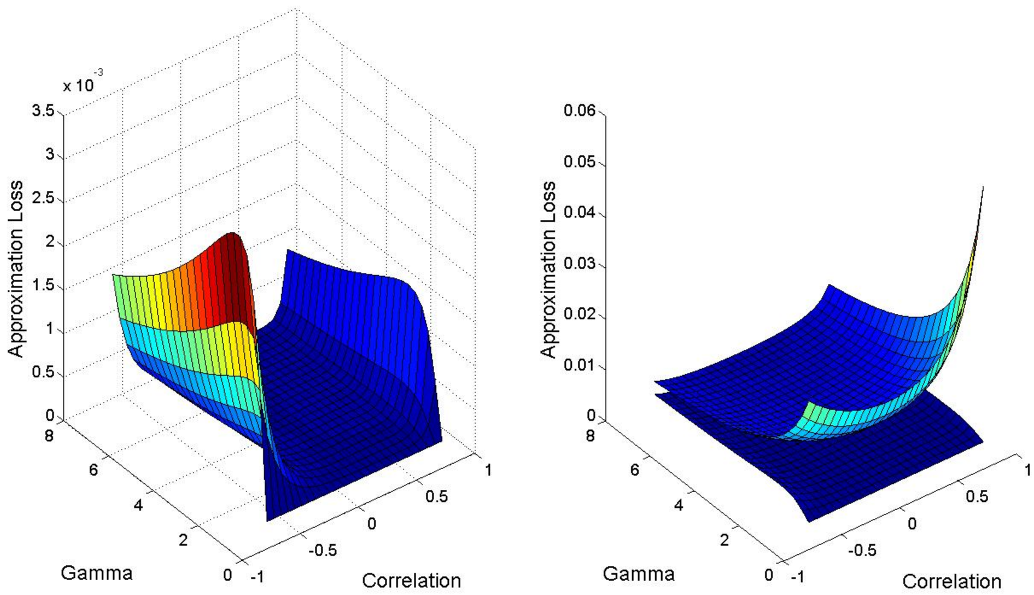

Figure 8: $\gamma > \theta$, correlation between -0.8 and 0.8. **Left:** The approximation loss in a market with two binomially distributed assets. The expected excess return for asset 1 is 5.75% and 4.5% for asset 2, whereas the standard deviation of the excess return is 16.25% for the first asset and 22.5% for the second asset. The approximation loss increases with increasing absolute value of correlation. **Right:** The upper surface depicts the approximation loss for the $1/n$ strategy in a market with two binomially distributed assets. The lower surface gives the approximation loss for our approximation strategy, (Figure 8, left). The $1/n$ strategy is clearly outperformed by our approximation approach for many combinations of asset correlation and relative risk aversion.

Garlappi / Uppal (2009), the risky fund can be composed according to the $1/n$ rule without much of an effect given parameter uncertainty. We ignore this uncertainty. The investor only decides how much money to allocate to the risk-free asset. Hence the portfolio problem is

$$\max_{\hat{\alpha}} \mathbb{E}\left[\left(1+\frac{\hat{\alpha}'\mathbf{r}}{\gamma}\right)^{1-\gamma}\right] \; s.t.\; \hat{\alpha}_1 = \ldots = \hat{\alpha}_n.$$

We measure the loss of the $1/n$ portfolio approach against the unrestricted optimal portfolio. Figure 8, right, shows the approximation loss for the $1/n$-policy for the market setting in which both loans have the same parameters as in Figure 8, left. It indicates large approximation losses in the range of 2% to 6%. The structure of the $1/n$-portfolio clearly differs from that of the optimal portfolio, triggering also a volume effect. This finding does not invalidate that of DeMiguel, Garlappi and Uppal (2009) which is driven by uncertainty about the parameters of the multivariate return distribution.

6 Conclusion

HARA-utility functions cover a wide spectrum of utility functions with declining, constant and increasing relative risk aversion. We constrain our analysis to declining absolute risk aversion and ask whether the parameters of the HARA-utility function really matter for optimal investment decisions. Changing the exponent of the utility function generates a volume effect: The amount invested in all risky assets together changes. We also obtain a structure effect: The structure of the risky portfolio changes with the exponent. The paper presents a simple mechanical rule to approximate the optimal portfolio by using a utility function with constant relative risk aversion which is smaller than that of the investor. It turns out that the approximation quality is surprisingly good in many market settings. In particular, the structure effect is mostly very small. This implies that an investor trading stocks can simply buy the market portfolio and the risk-free asset without noticeable harm whenever her relative risk aversion exceeds the constant elasticity of the pricing kernel of the market portfolio. Otherwise, the investor may buy the benchmark portfolio, which is a transformed market portfolio. with low constant elasticity of the pricing kernel.

Critical for a good approximation quality is that the higher non-centered moments of the optimal portfolio excess return relative to the second moment converge fast to zero. This can be assumed whenever the investor's relative risk aversion is higher than that used for the approximation and when the market setting rules out approximate arbitrage opportunities. We check for these opportunities by analyzing assets with strongly correlated binomial returns. Even then, the approximation quality is bad for very strong correlations only.

Further research may strengthen our understanding of the limits to the proposed approximation by using other multivariate probability distributions of asset returns and other utility functions.

A Proofs

Proof of Lemma 3:

a) $\gamma \geq \theta$:

Let p be the changing distribution parameter. At an intersection point $R := R(p) \in \{R^{(1)}(p); R^{(2)}(p)\}$, $\hat{V}^+(R) = \hat{V}^-(R)$ holds. Then, we have for an intersection point, using equation (20),

$$
\begin{aligned}
& \frac{\partial \hat{V}^+(R)}{\partial p} = \frac{\partial \hat{V}^-(R)}{\partial p} \\
\Leftrightarrow \quad & \frac{\theta}{\gamma} \exp\{a(\gamma)\} R^{(\theta/\gamma)-1} \frac{\partial R}{\partial p} + \frac{\partial a(\gamma)}{\partial p} \hat{V}^+(R) = \frac{\theta}{R_f} \frac{\partial R}{\partial p} \\
\Leftrightarrow \quad & \frac{\theta}{\gamma} \hat{V}^+(R) \frac{\partial \ln R}{\partial p} + \frac{\partial a(\gamma, p)}{\partial p} \hat{V}^+(R) = \overbrace{\left[\frac{\theta}{R_f} R + (\gamma - \theta) \right]}^{= \hat{V}^-(R) = \hat{V}^+(R)} \frac{\partial \ln R}{\partial p} - (\gamma - \theta) \frac{\partial \ln R}{\partial p} \\
\Leftrightarrow \quad & \frac{(\theta - \gamma) \quad \overbrace{[\hat{V}^+(R) - \gamma]}^{= \hat{V}^-(R) - \gamma = \frac{\theta}{R_f} R - \theta}}{\gamma \hat{V}^+(R)} \frac{\partial \ln R}{\partial p} = - \frac{\partial a(\gamma)}{\partial p} \\
\Leftrightarrow \quad & \frac{\partial \ln R}{\partial p} = \frac{\partial a(\gamma)}{\partial p} \frac{R_f \gamma \hat{V}^+(R)}{\theta (\gamma - \theta)(R - R_f)}.
\end{aligned}
$$

Since $\hat{V}^+$ is always positive and $R^{(1)} - R_f < 0$ and $R^{(2)} - R_f > 0$, it follows that $\frac{\partial \ln R^{(1)}}{\partial p}$ and $\frac{\partial \ln R^{(2)}}{\partial p}$ have opposite signs.

b) $\gamma < \theta$:

Using equation (21), the result follows by the same type of analysis.

Proof of Lemma 4:

The budget constraint is:

$$
\begin{aligned}
& \mathbb{E}^Q[\hat{V}^+(R_M)] = \mathbb{E}^Q[\hat{V}^-(R_M)] = \gamma \\
\Leftrightarrow \quad & \mathbb{E}^Q \left[\exp\{a(\gamma)\} R_M^{\theta/\gamma} \right] = \mathbb{E}^Q \left[\hat{V}^-(R_M) \right] = \gamma \\
\Rightarrow \quad & \int_0^\infty \hat{V}^+(y) \frac{\partial f^Q(y)}{\partial p} dy + \frac{\partial a(\gamma)}{\partial p} \gamma = \int_0^\infty \hat{V}^-(y) \frac{\partial f^Q(y)}{\partial p} dy
\end{aligned}
$$

Proof of Lemma 5:

The first order conditions, divided by the respective expected marginal utility, are:

$$
\mathbb{E} \left[\frac{r \left(1 + \hat{\alpha}^+ r/\gamma\right)^{-\gamma}}{\mathbb{E}\left[\left(1 + \hat{\alpha}^+ r/\gamma\right)^{-\gamma}\right]} \right] = \mathbb{E}[r] + \text{cov}\left(r, \frac{\left(1 + \hat{\alpha}^+ r/\gamma\right)^{-\gamma}}{\mathbb{E}\left[\left(1 + \hat{\alpha}^+ r/\gamma\right)^{-\gamma}\right]} \right) = 0,
$$

and

$$\mathbb{E}\left[\frac{r\,(1+\hat{\alpha}^{-}r/\phi)^{-\phi}}{\mathbb{E}\left[(1+\hat{\alpha}^{-}r/\phi)^{-\phi}\right]}\right] = \mathbb{E}[r] + \text{cov}\left(r, \frac{(1+\hat{\alpha}^{-}r/\phi)^{-\phi}}{\mathbb{E}\left[(1+\hat{\alpha}^{-}r/\phi)^{-\phi}\right]}\right) = 0.$$

$\hat{\alpha}^{+} > [=](<)\hat{\alpha}^{-}$ if and only if:

$$\mathbb{E}\left[r\frac{(1+\hat{\alpha}^{-}r/\gamma)^{-\gamma}}{\mathbb{E}[(1+\hat{\alpha}^{-}r/\gamma)^{-\gamma}]}\right] > [=](<)\ 0 = \mathbb{E}\left[r\frac{(1+\hat{\alpha}^{-}r/\phi)^{-\phi}}{\mathbb{E}\left[(1+\hat{\alpha}^{-}r/\phi)^{-\phi}\right]}\right].$$

Subtracting $\mathbb{E}[r]$ on both sides proves the Lemma.

B Characteristics of the Δ function

$$\Delta(\alpha r) = \frac{(1+\alpha r/\gamma)^{-\gamma}}{\mathbb{E}\left[(1+\alpha r/\gamma)^{-\gamma}\right]} - \frac{(1+\alpha r/\phi)^{-\phi}}{\mathbb{E}\left[(1+\alpha r/\phi)^{-\phi}\right]}.$$

Suppose $\gamma > \phi$. Then, for $\phi + \alpha r \to 0$, $(1+\alpha r/\phi)^{-\phi} \to \infty$ so that $\Delta(\alpha r) \to -\infty$. For $\alpha r \to \infty$, $\Delta(\alpha r) \to 0$ from below because

1. $(1+\alpha r/\gamma)^{-\gamma} << (1+\alpha r/\phi)^{-\phi}$ so that $\Delta(\alpha r) < 0$ and
2. $(1+\alpha r/\gamma)^{-\gamma}$ and $(1+\alpha r/\phi)^{-\phi}$ converge to zero.

Since $\mathbb{E}[\Delta(\alpha r)] = 0$, $\Delta(\alpha r)$ needs to be positive in some range of αr. Hence, $\Delta(\alpha r)$ has at least one maximum and one minimum. At an extremum,

$$\frac{\partial \Delta}{\partial(\alpha r)} = 0 \Leftrightarrow \frac{(1+\alpha r/\gamma)^{-\gamma-1}}{\mathbb{E}\left[(1+\alpha r/\gamma)^{-\gamma-1}\right]} = \frac{(1+\alpha r/\phi)^{-\phi}}{\mathbb{E}\left[(1+\alpha r/\phi)^{-\phi}\right]}$$

$$\Leftrightarrow \left(1+\frac{\alpha r}{\gamma}\right) = c\left(1+\frac{\alpha r}{\phi}\right)^{\frac{\phi+1}{\gamma+1}},$$

with $c > 0$. The left hand side of this equation is linear in αr while the right hand side is strictly concave. Therefore, at most two values of αr satisfy this equation. Hence, $\Delta(\alpha r)$ has one maximum and one minimum.

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