Pricing CDOs with Correlated Variance Gamma Distributions

Thomas Moosbrucker*

First Version: October 2005

This Version: January 2006

*Department of Banking, University of Cologne, Albertus-Magnus-Platz, 50923 Köln, Germany. Phone: +49 (0)221 470 3967, Fax: +49 (0)221 470 7700, email: Moosbrucker@wiso.uni-koeln.de

Pricing CDOs with Correlated Variance Gamma Distributions

Abstract

In this article, we propose a method for synthetic CDO pricing with Variance Gamma processes and distributions. First, we extend a structural model proposed by Luciano and Schoutens [2005] by allowing a more general dependence structure. We show that our extension leads to a correlation smile as observed in liquid index tranches. Since this method is not adequate for practical purposes, we extract the dependence structure into a factor approach based on Variance Gamma distributions. This approach allows for an analytical solution for the portfolio loss distribution. The model fits to prices of liquid CDS index tranches. It can be used to price bespoke CDOs in a consistent way.

1 Introduction

In the 1980s, Collateralized Debt Obligations (CDOs) were introduced for balance sheet risk management. The emergence of credit derivatives in the 1990s offered the possibility of synthetic risk transfer of a portfolio of bonds or loans, too. Since 2003, credit risk of standardised portfolios is traded in a liquid market in the CDS indices iBoxx and Trac-X. These indices merged into iTraxx in 2004. Standardised tranches that are linked to these indices started to be actively quoted. Thus, the entire distribution of portfolio loss (as seen by market participants) became an observable variable.

This development poses new challanges to credit risk models. One should expect the most common credit risk models to match the market implied loss distribution. However, this is not true. Using the common one factor Gaussian copula approach, different correlation parameters are needed to price different tranches. Thus, the dependence structure of defaults is not Gaussian.

As an alternative, we propose Variance Gamma (VG) processes and distributions for pricing liquid CDS index tranches. The following section briefly describes the Gaussian copula approach and the problems related to this method. We give a survey over the possible solutions to these problems in the literature and motivate the approach of this article. Section 3 extends a structural model proposed by Luciano and Schoutens [2005]. The abilities of this model in explaining the dependence structure implied by liquid tranches of DJ iTraxx are examined. In section 4, we propose a factor copula approach that replicates the dependence structure of the structural model and that is analytically tractable. Section 5 concludes. Appendix A gives the results concerning VG processes and distributions. We postpone proofs to Appendix B.

2 Valuation of CDOs

A Collateralized Debt Obligation is a securitisation of a portfolio of bonds or loans. The underlying portfolio is transfered to a Special Purpose Vehicle that issues securities on the portfolio in several tranches. Each tranche is defined by an attachment point L_a and a detachment point L_d. For a percentual loss of $L_{\text{portfolio}}$ of the underlying portfolio, the tranche suffers a percentual loss of

$$L_{\text{tranche}} = \max\{\min(L_{\text{portfolio}}, L_d) - L_a, 0\}.$$

The lowest tranche has $L_a = 0$ and is called the equity tranche. Since it already suffers from the first loss in the portfolio, it is the riskiest tranche and has to pay the highest spread to investors. For attachment points between 3 and 7 percent, tranches are called mezzanine, while the highest tranches are called senior or super-senior.

In synthetic CDOs, portfolio credit risk is transfered via Credit Default Swaps. With these instruments, not all tranches need to be sold to investors. On the basis of standardised tranches two parties agree to act as protection buyer and protection seller for this particular single tranche. Standard portfolios exist in the indices DJ CDX NA for entities in Northern America and DJ iTraxx for European entities.[1] The main indices consist of 125 equally weighted entities. The attachment and detachment points are 0%, 3%, 7%, 10%, 15% and 30% for CDX NA and 0%, 3%, 6%, 9%, 12% and 22% for DJ iTraxx. There is also the possibility to trade the whole index. This corresponds to a tranche with $L_a = 0\%$ and $L_d = 100\%$. Spreads are quoted in basispoints per year for all tranches. The only exception is the equity tranche, where spread is quoted as a percentage upfront payment plus 500 bp running premium. Table 1 shows market quotes of DJ iTraxx on June 24, 2005.

Tranche	$0-3\%$	$3-6\%$	$6-9\%$	$9-12\%$	$12-22\%$	Index
Spread	30.0%	98bp	34bp	20bp	14bp	40.0bp

Table 1: Market quotes of DJ iTraxx 5 year on June 24, 2005. Spread of the equity tranche is quoted as a percentage upfront plus 500bp running premium. The other tranches are quoted as bp per year. Source: Nomura Fixed Income Research.

2.1 One Factor Gaussian Copula

The Gaussian copula model has become the standard market model for valuing synthetic CDOs. In its basic form, for every entity i in the portfolio a standard normal random

[1] See Amato and Gyntelberg [2005] for detailed descriptions of these indices.

variable X_i is defined by

$$X_i \;=\; \sqrt{\rho}M + \sqrt{1-\rho}Z_i. \tag{1}$$

M and Z_i are standard normally distributed and $|\rho| \leq 1$. The factor M represents a systematic and Z_i an idiosyncratic risk factor of entity i. For $i \neq j$, the correlation of X_i and X_j is given by ρ. Entity i defaults, if X_i is smaller than some default threshold C_i. The default threshold is determined so that the risk neutral default probability[2] $Q_i(\tau)$ of entity i for every time τ is given by $Q_i(\tau) = \Phi^{-1}(C_i)$, where Φ is the cumulative distribution function of a standard normal random variable. Then the distribution of the number of defaults can be obtained.[3] If one assumes constant recovery rates, this distribution implies a distribution of portfolio loss. By assuming a zero mark-to-market value for each tranche, spreads on the tranches can be calculated.

The main advantage of the model is the independence of defaults when conditioned to the common risk factor. This allows a simple implementation and fast computations. However, when we apply the model to liquid tranches of the credit indices DJ CDX or iTraxx, it fails to fit market prices of the tranches. Different correlation parameters are needed to fit the prices of different tranches. For equity and senior tranches this implied correlation is higher than for mezzanine tranches. This phenomenon is known as the correlation smile of implied correlation. Table 2 shows the implied correlations corresponding to the quotes of table 1.[4]

Tranche	$0-3\%$	$3-6\%$	$6-9\%$	$9-12\%$	$12-22\%$
implied correlation	0.196	0.054	0.119	0.173	0.315

Table 2: Implied correlations of DJ iTraxx 5 year on June 24, 2005.

Market prices of liquid index tranches are used to calibrate models for the valuation of bespoke CDOs. If a model prices all liquid tranches correctly using the same parameter set, then a bespoke CDO tranche with non-standard attachment and detachment points can

[2]The risk neutral default probablity can be determined from sinlge name credit default swaps.

[3]See e.g. Gibson [2004] for details.

[4]These implied correlations slightly depend on certain assumptions about the portfolio structure. We have assumed an infinitely large portfolio of identical entities.

be priced in a consistent manner. If this is not the case — as in the Gaussian framework — one needs to develop further techniques.

Within the Gaussian framework, one of these techniques consists of calculating base correlations. These are implied correlations of hypothetical equity tranches (i.e. tranches with attachment point $L_a = 0$) with varying detachment points. The advantage of base correlations over implied correlations is that they are monotonically increasing along with the detachment point. One can therefore interpolate between base correlations in order to price non-standard tranches.

However, this approach is rather an ad hoc method than a consistent way of CDO pricing. It does not resolve the fundamental inconsistency in using the Gaussian copula approach. The search for models that can price all tranches with using one single parameter set has therefore been an active field of research in recent time. In the next subsection, we give a short survey over the methods proposed so far.

2.2 Further Methods

An natural idea is to try other copulas than the Gaussian. The choices proposed so far include the student t, double t, Clayton and Marshall-Olkin copula. Burtschell, Gregory and Laurent [2005] provide an overview over the results of calibrating these copula approaches to market spreads of DJ iTraxx. They find that the double t copula fits the observed spreads best. This copula has a small numerical disadvantage: if the risk factors M and Z_i in (1) and t-distributed, then the distribution of X_i depends on ρ and has to be calculated numerically.

Recently, Kalemanova, Schmid and Werner [2005] proposed a factor copula approach based on Normal Inverse Gaussian distributions. They show that calibration to liquid index tranches is as good as by the double t copula. It was this idea that inspired us to the use of the VG copula in section 4.

Empirical studies of de Servigny and Renault [2004] and Das et al. [2004] show that default correlations increase in times of a recession. In the last months, several autors have proposed models that incorporate this fact. Andersen and Sindenius [2005] extend the Gaussian copula model as they correlate ρ and M in equation (1) negatively.

Hull, Predescu and White [2005] have developed a structural model where firm value processes are correlated Brownian motions. The degree of correlation may depend on the systematic part of the process and therefore on the state of the economy. When they assume a negative dependence of correlation and the systematic part of the firm value processes, the authors show that spreads of CDX NA and iTraxx are fitted significantly better than by a constant correlation.

Another attempt to generate a dependence structure as observed in the market consists of the introduction of a stochastic business time. This idea has been used for the valuation of equity derivatives for a long time. In a firm value approach, stochastic business time leads to varying volatilities of the firm value process. If business time goes fast, firm values vary more and are therefore more likely to hit the default barrier. Thus, a fast business time corresponds to a bad economic environment.

Giesecke and Tomecek [2005] model default times in a portfolio as times of jumps of a Poisson process. The time scale of this process is varied depending on incoming information like economic environment and defaults. In this way, contagion effects can be considered.

Joshi and Stacey [2005] use Gamma processes to calibrate an intensity model to market prices of liquid CDO tranches. When business time is the sum of two Gamma processes, they show that their model can fit to the correlation smile of DJ iTraxx.

Cariboni and Schoutens [2004] model firm value processes and use Brownian motions subordinated by Gamma processes. The resulting Variance Gamma processes (VG processes) are calibrated to single name credit curves. Luciano and Schoutens [2005] extend the approach of Cariboni and Schoutens [2004] for the valuation of default baskets. All firm value processes follow the same Gamma process and thus the same business time. For every entity in the portfolio, they model its firm value process as an exponential of a Variance Gamma process. The parameters of these VG processes are determined by the credit curves of the corresponding single name CDS. Since all firm value processes follow the same business time and since the Brownian motions are independent, the complete dependence structure is determined by these parameters.

The model we propose in this paper extends the approach of Luciano and Schoutens

[2005] by allowing a more general dependence structure. We have chosen Variance Gamma processes and their distributions for our model, since they have a number of good mathematical properties and since they have proven to explain a number of economic findings. Mathematically, the distributions have nice properties such as leptokursis and fait tails. Their densities are known in closed form and the class of VG distributions is closed under scaling and convolution if parameters are chosen suitably. Economically, Cariboni and Schoutens show that their model fits to a variety of single name credit curves. The idea of a stochastic business time leads to an increase of default correlations in recessions, which has proven to create correlation smiles. Finally, VG processes have shown to explain the volatility smile in equity options (see Madan et al. [1998]). Thus, Variance Gamma processes and distributions are a natural candidate for explaining the correlation smile.

3 The Structural Variance Gamma Model

We briefly describe the structural model of Luciano and Schoutens [2005]. This serves as a base case for three extensions in the following subsections. In the last subsection, we examine the ability of the model and its extensions to explain the observed correlation smile in liquid index tranches. We provide the properties of VG processes needed for this article in Appendix A.

3.1 Base Case: Identical Gamma Processes, independent Brownian Motions

Let N be the number of entities in the portfolio and for every $i \in \{1, \ldots, N\}$ let

$$X_t^{(i)} = \theta_i G_t + \sigma_i W_{G_t}^{(i)} \tag{2}$$

be a VG process with parameters $(\theta_i, \nu, \sigma_i)$. This means that $(G_t)_{t \geq 0}$ is a Gamma process with parameters (ν^{-1}, ν)[5] and for every i the process $(W_t^{(i)})_{t \geq 0}$ is a standard Brownian motion. For $i \neq j$, these Brownian motions are independent. The firm value process $(S_t^{(i)})_{t \geq 0}$ of entity i is given by

[5]The reason why the parameter ν does not depend on i will be explained in the next subsection.

$$S_t^{(i)} = S_0^{(i)} \cdot \exp\left(r_t t + X_t^{(i)} + \omega_i t\right). \tag{3}$$

In equation (3), r_t denotes the risk free interest rate at t and

$$\omega_i = \frac{1}{\nu}\log\left(1 - \frac{1}{2}\sigma_i^2\nu - \theta_i\nu\right)$$

a parameter to ensure the martingale property of the discounted firm value $\frac{S_t^{(i)}}{\exp(r_t t)}$ (see Appendix B for details). Entity i defaults at time $\tau > 0$ if

$$\tau = \min_{t\in T}\{S_t^{(i)} < L_t^{(i)}\}.$$

In this base case, all firm value processes (3) follow the same Gamma subordinator $(G_t)_{t\geq 0}$. Luciano and Schoutens argue that all firms are subject to the same economic environment and thus information arrival should affect business time of all entities. The Brownian motions are independent, however. This means that all correlation involved is caused by the common business time.

Figure 1 shows two paths of VG processes with identical Gamma processes and independent Brownian motions. Jumps occur at identical times, but their directions are conditionally independent.

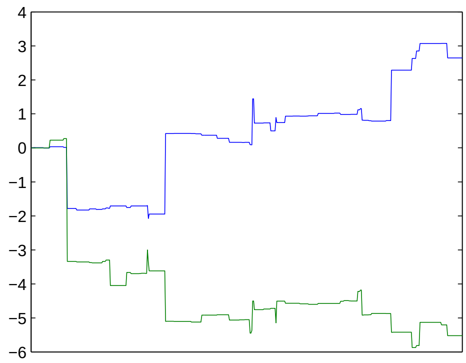

Figure 1: VG processes with identical Gamma processes

We extend this correlation structure in the following subsections. First, we allow business time to be correlated and not identical for all entities. We therefore allow changes in business time to be caused by systematic or idiosyncratic information. Second, we allow the Brownian motions to be dependent. This means that the directions of jumps are correlated. Finally, we integrate these two ideas into a third extension. In section 4 we show that the last extension may be solved analytically under some simplifying assumptions.

3.2 Extension A: Correlated Gamma Processes, independent Brownian Motions

In this extension, changes in business time may be caused by information concerning the entire economy or by information about the individual firm. If we decompose $(G_t)_{t\geq 0}$ into a systematic and an idiosyncratic part, we can incorporate this idea into the model. For every i, we seperate $(G_t^{(i)})_{t\geq 0}$ into a sum of two Gamma processes

$$\mathrm{d}G_t^{(i)} = \mathrm{d}F_t + \mathrm{d}U_t^{(i)}.$$

In this decomposition, $(F_t)_{t\geq 0}$ and $(U_t^{(i)})_{t\geq 0}$ are independent Gamma processes with parameters $(a_F, b_F) = (a\nu^{-1}, \nu)$ and $(a_{U^{(i)}}, b_{U^{(i)}}) = ((1-a)\nu^{-1}, \nu)$ with $0 \leq a \leq 1$. It follows that $(G_t^{(i)})_{t\geq 0}$ is a Gamma process with parameters $((1-a)\nu^{-1} + a\nu^{-1}, \nu) = (\nu^{-1}, \nu)$.[6]

For $a \to 1$, this extension coincides with the base case. The parameter a controls for the pairwise correlation $\mathrm{Corr}(X_1^{(i)}, X_1^{(j)})$. In Appendix B, we show that for independent Brownian motions $W^{(i)}$ and $W^{(j)}$ and for $i \neq j$ we have

$$\mathrm{Corr}(X_1^{(i)}, X_1^{(j)}) \quad = \quad a \cdot \frac{\theta_i \theta_j \nu}{\sqrt{\theta_i^2 \nu + \sigma_i^2}\sqrt{\theta_j^2 \nu + \sigma_j^2}}. \tag{4}$$

For a homogeneous portfolio (i.e. identical credit curves) all entities have identical parameters $\theta_i = \theta$, $\sigma_i = \sigma$ and all correlations in the interval $[0; \frac{\theta^2\nu}{\theta^2\nu+\sigma^2}]$ can be reached. The upper bound is obtained for $a \to 1$, which is the base case. The lower bound is reached for $a \to 0$, i.e. for independent Gamma processes.

One should bear in mind, however, that this correlation has to be treated with care. For example, if $\theta_i = \theta_j = 0$, the processes $X^{(i)}$ and $X^{(j)}$ are uncorrelated, but not necessarily

[6] For two independent Gamma variables $X \sim \Gamma(a_X, \lambda)$ and $Y \sim \Gamma(a_Y, \lambda)$ their sum is Gamma distributed with $X + Y \sim \Gamma(a_X + a_Y, \lambda)$. This is also the reason for which we chose the parameter ν to be identical for all i.

independent. For a better understanding of the dependence structure, copulas have to be regarded.

Figure 2 shows sample paths of correlated VG processes for $a = 0.5$. There are times where only one process has a large jump. These jumps are caused by the idiosyncratic factor $U^{(i)}$. At other times, both realisations jump. Those jumps are caused by the systematic factor F.

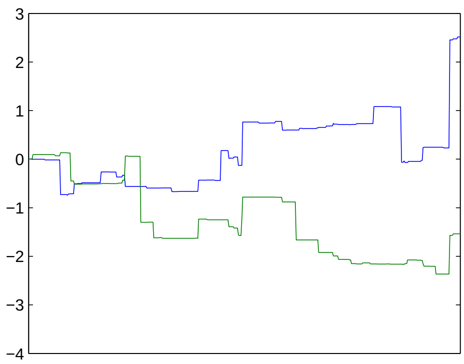

Figure 2: VG processes with correlated Gamma processes

3.3 Extension B: Identical Gamma Processes, dependent Brownian Motions

The extension proposed in the last subsection correlates times of high activity of the VG processes. The directions of these jumps are conditionally independent since the Brownian motions $W^{(i)}$ and $W^{(j)}$ are independent. If jumps are caused by information concerning the state of the economy, this information should be (more or less) good or bad for all companies. The model reflects this fact if the Brownian motions are correlated.

The correlation of the Brownian motions is modelled via

$$\mathrm{d}W_t^{(i)} = \sqrt{b}\mathrm{d}M_t + \sqrt{1-b}\mathrm{d}Z_t^{(i)}.$$

In this extension, we choose the same Gamma subordinator for each entity. Then, for the correlation of the VG processes we find for $i \neq j$ (see Appendix B)

$$\mathrm{Corr}(X_1^{(i)}, X_1^{(j)}) \quad = \quad \frac{\theta_i \theta_j \nu + \sigma_i \sigma_j b}{\sqrt{\theta_i^2 \nu + \sigma_i^2}\sqrt{\theta_j^2 \nu + \sigma_j^2}}. \tag{5}$$

For a homogeneous portfolio with identical parameters $\theta_i = \theta$, $\sigma_i = \sigma$ all correlations in the interval $[\frac{\theta^2\nu}{\theta^2\nu+\sigma^2}, 1]$ can be reached, if positive values for b are considered. The lower bound corresponds to $b = 0$. If $b = 1$, the Variance Gamma processes are identical and therefore have a correlation of 1.

Figure 3 shows two paths for $b = 0.5$. Large jumps occur at identical times and their directions are correlated.

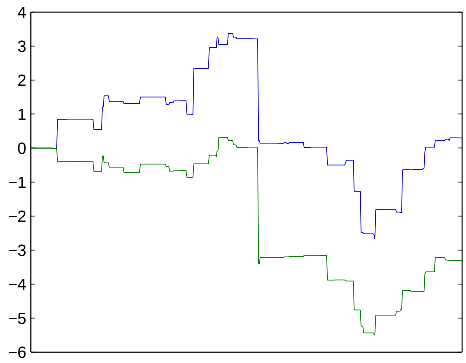

Figure 3: VG processes with identical Gamma subordinators and correlated Brownian motions

3.4 Extension C: Sum of two independent VG Processes

While in the above extensions we decompose either the Gamma or the Brownian part of the VG process, we now decompose the whole VG process into two VG processes. We restrict ourselves to the case where all VG processes $(X_t^{(i)})_{t\geq 0}$ have the same parameters (θ, ν, σ).

We choose $0 < c < 1$ and decompose $(X_t^{(i)})_{t\geq 0}$ into

$$X_t^{(i)} = cM_t + \sqrt{1-c^2}Z_t^{(i)} \tag{6}$$

with

$$(\theta_M, \nu_M, \sigma_M) = \left(c\theta, \frac{\nu}{c^2}, \sigma\right), \tag{7}$$

$$(\theta_{Z^{(i)}}, \nu_{Z^{(i)}}, \sigma_{Z^{(i)}}) = \left(\sqrt{1-c^2}\theta, \frac{\nu}{(1-c^2)}, \sigma\right). \tag{8}$$

From the results of Appendix A it is clear that the sum $(X_t^{(i)})_{t\geq 0}$ is again a VG process and its parameters are given by

$$(\theta_{X^{(i)}}, \nu_{X^{(i)}}, \sigma_{X^{(i)}}) = (\theta, \nu, \sigma).$$

We show in Appendix B that

$$\mathrm{Corr}(X_1^{(i)}, X_1^{(j)}) = c^2 \tag{9}$$

for $i \neq j$.

Note that this extension is not the junction of extensions A and B for $a = b = c^2$. Nevertheless, the correlation of $(X_t^{(i)})_{t\geq 0}$ and $(X_t^{(j)})_{t\geq 0}$ stems from both Gamma and Brownian correlation.

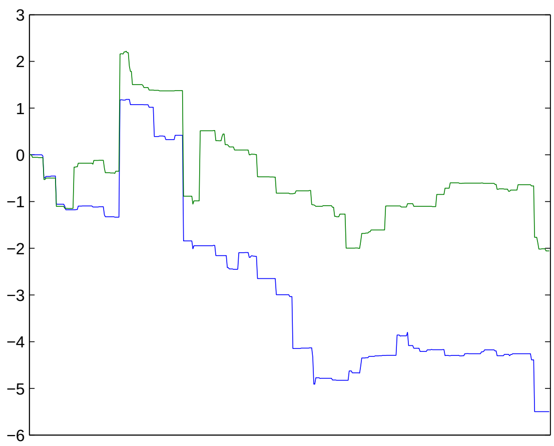

Figure 4: Correlated VG processes

Figure 4 shows two paths of correlated VG processes for $c^2 = 0.5$. Jumps occur for systematic and idiosyncratic reasons. The directions of systematic jumps are correlated.

Since all firm value processes possess identical parameters, the entities in the portfolio are assumed to have identical credit curves. This simplification and the fact that all processes are of VG type allow for an analytical solution that is given in the next section.

3.5 Application to Index Tranches

We now apply the model to data on tranches of DJ iTraxx 5 year from June 24, 2005 to December 9, 2005. For each of these dates, we also have index spreads of DJ iTraxx for the maturities of 3, 5, 7 and 10 years. Our data set consists of 25 weekly quotes. The index spread curves are used to determine the parameters $(\theta_i, \nu, \sigma_i)$. Since our data set does not include credit curves for the single entities, we assume that the portfolio is homogeneous with respect to default probabilities. Thus all processes have identical parameters.

For each date, the calibration is accomplished in two steps: First, we determine the parameters (θ, ν, σ) to fit the index spread curve given by quotes of the indices with different maturities. In the base case, the entire correlation structure is determined and we calculate tranche spreads. In our extensions, we conduct a second step: we determine the correlation paramters a, b and c such that the spread of the equity tranche is matched.

In all the computations of this subsection, we have set the initial firm value to $S_0 = 100$, the default barrier to $L = 50$ and recovery rates to $R = 40\%$. We simulated the firm value process on a discrete time grid with $\mathrm{dt} = 0.02$.[7] The risk-free zero curve is the EUR zero curve.

We calculate par spreads in the usual way. We assume that all payments (fee and contingent legs) occur on quarterly payment dates T_i, $i = 1, \ldots, n$. The expected default loss $\mathrm{EL}(T_i)$ on the tranche up to payment date T_i is given by

$$\mathrm{EL}(T_i) = \int_0^1 \max\left(\min(x(1-R), L_d) - L_a, 0\right) f_{\mathrm{Loss}(T_i)}(x)\mathrm{d}x,$$

where R is the (constant) recovery rate, L_a and L_d denote the tranche's attachment and detachment points and $f_{\mathrm{Loss}(T_i)}$ is the probability density function of losses until T_i.

The tranche's mark-to-market (from the investors view) for a spread per annum of s

[7] We have tested the model to robustness with respect to the assumptions on L, R and dt. Qualitatively, the results were the same, altough computation speed obviously depends on dt.

is then given by[8]

$$\begin{aligned}\mathrm{MTM}(s) &= \mathrm{Fee} - \mathrm{Contingent} \\ &= s \cdot \sum_{i=1}^{n} \Delta_i e^{rT_i} \left((L_d - L_a) - \mathrm{EL}(T_i) \right) \\ &\quad - \sum_{i=1}^{n} e^{-rT_i} \left(\mathrm{EL}(T_i) - \mathrm{EL}(T_{i-1}) \right),\end{aligned}$$

where Δ_i is the accrual factor for payment date i. The par spread is obtained by $\mathrm{MTM}(s_{\text{par}}) = 0$.

Table 3 shows the results of the calibration to the index spread curve. Calibration was done to minimise the absolute pricing error (APE). In this calibration, the APE denotes

$$APE = \sum_{\text{maturities}} \left| \text{spread}^{\text{Market}}_{\text{Maturity}} - \text{spread}^{\text{Model}}_{\text{Maturity}} \right|.$$

We have calibrated the full model to spread curves as well as some cases, where we restricted either to $\theta = 0$ or $\nu = 1$. If $\theta = 0$, we get unskewed distributions. If $\nu = 1$, the Gamma distributions are exponential distributions. We find that all cases lead to a good fit to observed spreads. This result confirms the findings of Cariboni and Schoutens [2004].

Maturity	3 years	5 years	7 years	10 years	average APE
Market	21.87	37.00	46.68	57.31	
restr. to $\theta = 0$	22.60	36.27	49.54	66.23	13.48
	(0.92)	(0.77)	(2.87)	(8.93)	
restr. to $\nu = 1$	22.84	36.27	46.63	61.15	6.18
	(0.99)	(0.70)	(0.28)	(3.88)	
unrestr.	25.18	37.12	46.39	57.08	5.55
	(3.31)	(0.53)	(0.82)	(0.88)	

Table 3: Calibration to spread curves of DJ iTraxx from June 24, 2005 to December 9, 2005. Numbers in parantheses denote average pricing errors. Market quotes are mid quotes obtained by International Index Company.

[8]See for example Gibson [2004] for details.

Given the parameters from this calibration, we can calculate model implied tranche spreads in the base case and the extensions. In the base case, there is no additional correlation parameter, so tranche spreads are already determined by the values of (θ, ν, σ). We find that in all cases, the spread of the equity tranche is significantly underestimated. This shows that implied correlation of this approach is too high. Thus, we can use extension A and C to calibrate the correlation parameters a and c to match the equity tranche. We have determined a and c such that the spread of the equity tranche is matched. For comparison of the over all fit of the models, we have included the average APE into the table, where the APE is given by

$$APE \quad = \sum_{\text{tranches}} \left| \text{spread}^{\text{Market}}_{\text{Tranche}} - \text{spread}^{\text{Model}}_{\text{Tranche}} \right| . \tag{10}$$

Since extension B further increases correlation compared to the base case, we cannot calibrate this extension to match the equity tranche.

We show the calibration results of the base case and extensions A and C to tranche spreads in table 4. For comparison, we have also included the values for the double t copula with 4 and 5 degrees of freedom. Burtschell et al [2005] find the double t(4) copula to match tranches best when compared to several other copulas.

We find that in the base case, the model spread of the equity tranche is much lower than observed in the market. The negative spread of -12.8 for the case restricted to $\nu = 1$ indicates that the model implied spread is even smaller than the 500bp running premium. In the extensions, the upfront premium of the equity tranche can be matched. In extension A, pricing errors are smaller than for the Gaussian Copula, which indicates a model implied correlation smile. APEs are still quite large compared to the double t(4) copula for $\theta = 0$ and in the unrestricted case. The best fit is achieved in the case restricted to $\nu = 1$. Extension C leads to a fit that is comparable to the double t(4) copula in all three cases.

	$0-3\%$	$3-6\%$	$6-9\%$	$9-12\%$	$12-22\%$	average APE
Market	27.1%	86.7	26.8	14.1	8.4	—
Gauss	27.1%	193.3	45.3	12.6	1.7	136.4
	—	(106.7)	(18.6)	(4.4)	(6.8)	
double t(4)	27.1%	97.1	36.4	21.1	10.8	33.0
	—	(10.4)	(9.6)	(7.1)	(2.4)	
double t(5)	27.1%	112.9	38.5	20.3	9.1	45.4
	—	(26.2)	(11.7)	(6.0)	(0.6)	
VG Base case, $\theta = 0$	17.0%	281.8	88.6	23.5	1.7	—
	(10.0)	(195.1)	(61.8)	(9.4)	(6.7)	
VG Base case, $\nu = 1$	-12.8%	143.3	115.1	97.6	73.7	—
	(39.8)	(56.6)	(88.3)	(83.5)	(65.3)	
VG Base case, unrest.	6.3%	278.5	153.4	89.8	32.3	—
	(20.9)	(192.0)	(127.3)	(76.5)	(24.3)	
VG Ext. A, $\theta = 0$	27.1%	182.4	42.1	8.5	0.5	124.7
	—	(95.8)	(15.3)	(5.6)	(7.9)	
VG Ext. A, $\nu = 1$	27.1%	106.1	32.7	15.9	5.9	38.4
	—	(24.5)	(8.4)	(3.0)	(2.6)	
VG Ext. A, unrest.	27.1%	150.9	54.0	24.5	6.5	104.2
	—	(64.2)	(27.2)	(10.4)	(2.0)	
VG Ext. C, $\theta = 0$	27.1%	91.1	31.1	17.5	8.4	24.9
	—	(13.5)	(6.2)	(3.8)	(1.4)	
VG Ext. C, $\nu = 1$	27.1%	102.5	32.3	16.3	6.8	32.9
	—	(20.4)	(7.5)	(3.3)	(1.7)	
VG Ext. C, unrest.	27.1%	98.5	36.7	21.3	10.4	37.8
	—	(17.7)	(10.9)	(7.2)	(2.1)	

Table 4: Average Market spreads and average model implied spreads for DJ iTraxx 5yr between June 24, 2005 and December 9, 2005. Numbers in parantheses denote average pricing errors.

4 VG Copula Model

We showed in the last section that the correlation smile in liquid index tranches may be explained by the structural variance gamma model. For practical purposes however, there are some disadvantages of the structural model. First, we have to use Monte Carlo simulations to calibrate the model to market quotes, which leads to slow implementations. Furthermore, we have used the parameters (θ, ν, σ) solely for the calibration to the index spread curve and not for the calibration to tranche prices.

We therefore implement the idea of extension C above into a factor copula approach. We assume a homogeneous portfolio and constant default intensities for the entities. This default intensity is determined by the 5 year index spread. Thus, we can use the parameters of the VG distribution and a correlation parameter to calibrate the approach to tranche quotes. Since VG distributions possess a fourth parameter μ in general (see Appendix A) and we restrict our distributions to have zero mean and unit variance, this gives us 2 degrees of freedom besides correlation.

Another assumption used in our examples is the portfolio to consist of an infinite number of entities. This allows us to use the large homogeneous portfolio (LHP) approximation[9], which is a common procedure for other copula approaches. As with the Gaussian copula, this assumption may be relaxed. In this case, a semi-analytical approach similar to the ones for the Gaussian copula has to be conducted. However, the LHP method allows us to calculate an analytical solution for the portfolio loss distribution.

4.1 VG Copula

In analogy to the Gaussian Copula we define the one factor VG copula by

$$X_i = cM + \sqrt{1-c^2}Z_i, \tag{11}$$

[9]For the Gaussian copula model, this approximation was introduced by Vasicek [1987].

where M and Z_i are independently VG distributed random variables. The distribution parameters are given by[10]

$$\begin{aligned} M &\sim VG\left(c\theta, \frac{\nu}{c^2}, \sqrt{1-\nu\theta^2}, -c\theta\right), \\ Z_i &\sim VG\left(\sqrt{1-c^2}\theta, \frac{\nu}{1-c^2}, \sqrt{1-\nu\theta^2}, -\sqrt{1-c^2}\theta\right). \end{aligned}$$

Note that the distributions of M and Z_i correspond to the distributions of M_1 and $Z_1^{(i)}$ in (7) and (8) in section 3.4 with $\sqrt{1-\nu\theta^2}$ and $\mu = -\theta$ to obtain zero mean and unit variance. Using the results about scaling and convoluting VG variables at the end of Appendix A, we find

$$\begin{aligned} cM &\sim VG\left(c^2\theta, \frac{\nu}{c^2}, c\sqrt{1-\nu\theta^2}, -c^2\theta\right), \\ \sqrt{1-c^2}Z_i &\sim VG\left((1-c^2)\theta, \frac{\nu}{1-c^2}, \sqrt{1-c^2}\sqrt{1-\nu\theta^2}, -(1-c^2)\theta\right). \end{aligned}$$

The distribution of X_i is

$$X_i = cM + \sqrt{1-c^2}Z_i \sim VG\left(\theta, \nu, \sqrt{1-\nu\theta^2}, -\theta\right).$$

The third and fourth parameters were chosen such that all variables have zero mean and unit variance. We show in Appendix B that the correlation of X_i and X_j for $i \neq j$ is given by

$$\mathrm{Corr}(X_i, X_j) = c^2. \tag{12}$$

Using (11) and the LHP approximation we can deduce the loss distribution of the portfolio. If we denote the (common) default threshold by C and condition on the systematic factor M, then entity i defaults if

$$Z_i < \frac{C - cM}{\sqrt{1-c^2}}.$$

The conditional default probability is given by

$$P(X_i < C | M = m) = F_{Z_i}\left(\frac{C - cm}{\sqrt{1-c^2}}\right),$$

[10]See Appendix A for details on VG distributions and their properties as far as we need them for this article.

where F_{Z_i} denotes the cumulative distribution function of Z_i, which is identical for all i. We find that the loss distribution function of the portfolio is

$$F_{\text{portfolio loss}}(x) = F_M\left(\frac{\sqrt{1-c^2}F_{Z_i}^{-1}(x) - C}{c}\right), \tag{13}$$

if the recovery rates are zero. The proof from Vasicek [1987] applies here, too. We provide it in Appendix B.

4.2 Application to CDS Index Tranches

In this subsection, we calibrate the VG Copula model to the market quotes we have used in section 3.5. The default threshold C is determined to match the index spread. We can thus use the parameters θ, ν and c to calibrate the model to tranche spreads. The calibration is done by a minimisation of the absolute princing error

$$APE = \sum_{\text{tranches}} \left|\text{spread}_{\text{Tranche}}^{\text{Market}} - \text{spread}_{\text{Tranche}}^{\text{Model}}\right|.$$

while keeping the equity spread fixed to match the market spread. As in section 3.5, we assume a constant recovery rate of 40% for all entities. The risk-free zero curve is the EUR zero curve. First, we fix $\theta = 0$ which leads to unskewed distributions. In a second stage, we calibrate over all parameters. The results are given in table 5.

Both the restricted ($\theta = 0$) and the unrestricted case lead to a better fit than the double t(4) copula. The fit is slightly improved in the unrestricted case. In the unrestricted case, the average APEs of all tranches except the $[3-6\%]$-tranche are below 4bp, which is about the bid ask spread on a typical day.

Comparing the absolute pricing errors of table 4 (structural model) to those of table 5 (factor copula), we find that we could reduce this pricing errors by this analytical model. This is possible since we assumed a constant default intensity and ignored the CDS indices with maturities different than 5 years. One should not compare parameter values of ρ, θ and ν, since we have restricted ourselves to unit variances for all distributions in this section. Further differences occur since defaults are not triggered by a constant default barrier L as in the last section.

	0-3%	3-6%	6-9%	9-12%	12-22%	average APE	$\overline{\rho}$
Average Market Quotes	27.1%	86.7	26.8	14.1	8.4	—	
Gauss	27.1%	193.3	45.3	12.6	1.7	136.4	0.160
	—	(106.7)	(18.6)	(4.4)	(6.8)		
double t(4)	27.1%	97.1	36.4	21.1	10.8	33.0	0.197
	—	(10.4)	(9.6)	(7.1)	(2.4)		
double t(5)	27.1%	112.9	38.5	20.3	9.1	45.4	0.193
	—	(26.2)	(11.7)	(6.0)	(0.6)		
VG(ν)	27.1%	86.6	38.5	23.8	12.9	26.0	0.175
	—	(0.1)	(11.7)	(9.7)	(4.5)		
VG(ν,θ)	27.1%	75.4	26.3	14.3	6.5	20.7	0.208
	—	(11.2)	(3.6)	(1.6)	(2.3)		

Table 5: Calibration to DJ iTraxx 5yr Series 5 for weekly prices between June 24, 2005 and December 9, 2005. VG(ν) denotes the VG Copula model restricted to $\theta = 0$ and VG(ν,θ) the unrestricted model. Average pricing errors for the tranches are given inparantheses. The values of $\overline{\rho}$ denote correlation averages.

5 Conclusion

In this article, we proposed a valuation method for CDS index tranches by means of Variance Gamma processes and distributions. We showed that extensions of a structural model developed by Luciano and Schoutens [2005] can generate a correlation smile as observed in the market. Since computations within this model are time-consuming, we extracted the resulting dependence structure into a VG copula. This model shares the advantages of the one factor Gaussian copula of conditional independence and the LHP approximation. Yet, the VG Copula turns out to be more flexible and leads to a dependence structure that fits to observed tranche spreads. We can therefore price bespoke CDO tranches in a consistent way.

Appendix A: Variance Gamma processes and distributions

A random variable X is said to be Variance Gamma distributed *with parameters θ, ν, σ, μ ($X \sim VG(\theta, \nu, \sigma, \mu)$) if its density is given by*

$$f_X(x) = \frac{2\exp\left(\frac{\theta x}{\sigma^2}\right)}{\nu^{\frac{1}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{1}{\nu})}\left(\frac{(x-\mu)^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{1}{2\nu}-\frac{1}{4}} K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-\mu)^2\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right).$$

K is the modified Bessel function of the third kind,

$$K_\nu(x) = \frac{1}{2}\int_0^\infty y^{\nu-1}\exp\left(-\frac{1}{2}x(y+y^{-1})\right)dy.$$

The parameter domain is restricted to $\mu, \theta \in \mathbb{R}$ and $\nu, \sigma > 0$.[11]

Since this distribution is infinitely divisible, there exists a Lévy Process $(X_t)_{t\geq 0}$ such that X_1 has the above distribution. $(X_t)_{t\geq 0}$ is called Variance Gamma process and can be represented by

$$X_t = \mu \cdot t + \theta \cdot G_t + \sigma W_{G_t},$$

where $(G_t)_{t\geq 0}$ is a Gamma process parameters (ν^{-1}, ν) and $(W_t)_{t\geq 0}$ is a standard Brownian motion. In this article, we have set $\mu = 0$ when we consider VG processes.

If $X \sim VG(\theta, \nu, \sigma, \mu)$, the Laplace transform L_X of X is given by

$$L_X(z) \quad = \quad e^{\mu z}\left(1 - \theta\nu z - \frac{1}{2}\nu\sigma^2 z^2\right)^{-\frac{1}{\nu}}, z \in \mathbb{R}, \tag{14}$$

We therefore obtain the moments:

$$\begin{aligned}
\mathrm{E}[X] &= \mu + \theta, \\
\mathrm{Var}[X] &= \nu\theta^2 + \sigma^2, \\
\mathrm{S}[X] &= \theta\nu\frac{3\sigma^2 + 2\nu\theta^2}{(\sigma^2+\nu\theta^2)^{3/2}}, \\
\mathrm{K}[X] &= 3(1 + 2\nu - \nu\sigma^4(\nu\theta^2+\sigma^2)^{-2}).
\end{aligned}$$

Besides infinite divisibility, the class of VG distributions is closed under scaling and convolution if parameters are chosen suitably:

[11] This definition is taken from Bibby and Sørensen [2003] and we used the parameter transformation $\mu \to \mu$, $\alpha \to \sqrt{\frac{2}{\nu\sigma^2} + \frac{\theta^2}{\sigma^4}}, \beta \to \frac{\theta}{\sigma^2}, \lambda \to \frac{1}{\nu}$.

1. If $X \sim VG(\theta, \nu, \sigma, \mu)$ and $c > 0$, then

$$cX \sim VG(c\theta, \nu, c\sigma, c\mu).$$

2. If $X_1 \sim VG(\theta_1, \nu_1, \sigma_1, \mu_1)$ and $X_2 \sim VG(\theta_2, \nu_2, \sigma_2, \mu_2)$ such that $\sigma_1^2\nu_1 = \sigma_2^2\nu_2$ and $\frac{\theta_1}{\sigma_1^2} = \frac{\theta_2}{\sigma_2^2}$ then

$$X_1 + X_2 \sim VG\left(\theta_1 + \theta_2, \frac{\nu_1 + \nu_2}{\nu_1\nu_2}, \sqrt{\sigma_1^2 + \sigma_2^2}, \mu_1 + \mu_2\right).$$

Appendix B

Martingale property of $S_t^{(i)}$ in (3) in section 3.1

$$\begin{aligned} E\left[S_t^{(i)}\right] &= S_0^{(i)} \cdot \exp(rt + \omega_i t) \cdot E\left[\exp(X_t^{(i)})\right] \\ &= S_0^{(i)} \cdot \exp\left(rt + \frac{1}{\nu}\log\left(1 - \frac{1}{2}\sigma_i^2\nu - \theta_i\nu\right)t\right) \\ &\quad \cdot \exp\left(-\left(\frac{1}{\nu}\log\left(1 - \frac{1}{2}\sigma_i^2\nu - \theta_i\nu\right)t\right)\right) \\ &= S_0^{(i)} \cdot \exp(rt). \end{aligned}$$

The second equation holds as the characteristic exponent of $\left(X_t^{(i)}\right)_{t\geq 0}$ is given by (see Appendix A)

$$\Psi_{X^{(i)}}(u) = -\frac{1}{\nu}\log\left(1 + \frac{1}{2}u^2\sigma_i^2\nu - iu\theta_i\nu\right)$$

and since

$$E\left[\exp(X_t^{(i)})\right] = \exp\left(\Psi(-i) \cdot t\right).$$

Proof of (4) in section 3.2

If

$$\begin{aligned} X_t^{(i)} &= \theta_i G_t^{(i)} + \sigma_i W_{G_t^{(i)}}^{(i)}, \\ X_t^{(j)} &= \theta_j G_t^{(j)} + \sigma_j W_{G_t^{(j)}}^{(j)} \end{aligned}$$

as in section 3.2 then

$$\begin{aligned}
\mathrm{Cov}(X_t^i, X_t^j) &= \mathrm{E}\left(\prod_{k=i,j}\left(\theta_k(G_t^{(k)} - t) + \sigma_i\sqrt{G_t^{(k)}}W_t^{(k)})\right)\right) \\
&= \theta_i\theta_j\mathrm{Cov}\left(G_t^{(i)}, G_t^{(j)}\right) + \theta_i\sigma_j\mathrm{E}\left(G_t^{(i)}\sqrt{G_t^{(j)}}\right)\underbrace{\mathrm{E}(W_t^{(i)})}_{=0} \\
&\quad +\theta_j\sigma_i\mathrm{E}\left(G_t^{(j)}\sqrt{G_t^{(i)}}\right)\underbrace{\mathrm{E}(W_t^{(j)})}_{=0} \\
&\quad +\sigma_i\sigma_j\mathrm{E}\left(\sqrt{G_t^{(i)}G_t^{(j)}}\right)\underbrace{\mathrm{E}(W_t^{(i)})\mathrm{E}(W_t^{(j)})}_{=0} \\
&= \theta_i\theta_j\mathrm{Cov}\left(G_t^{(i)}, G_t^{(j)}\right) \\
&= \theta_i\theta_j\mathrm{Cov}\left(F_+U_t^{(i)}, F_t + U_t^{(j)}\right) \\
&= \theta_i\theta_j\mathrm{Var}(F_t) \\
&= \theta_i\theta_j a\nu t.
\end{aligned}$$

In the same way (or using the results of Appendix A) it can be shown that

$$\mathrm{Var}(X_t^{(i)}) = \left(\theta_i^2\nu + \sigma_i^2\right)\cdot t.$$

We therefore see that the correlation for all t is:

$$\mathrm{Corr}(X_t^{(i)}, X_t^{(i)}) = a\cdot\frac{\theta_i\theta_j\nu}{\sqrt{\theta_i^2\nu + \sigma_i^2}\sqrt{\theta_j^2\nu + \sigma_j^2}}.$$

Proof of (5) in section 3.3

If

$$\begin{aligned}
X_t^{(i)} &= \theta_i G_t + \sigma_i W_{G_t}^{(i)}, \\
X_t^{(j)} &= \theta_j G_t + \sigma_j W_{G_t}^{(j)}
\end{aligned}$$

as in section 3.3 then

$$\begin{aligned}
\operatorname{Cov}\left(X_t^{(i)}, X_t^{(j)}\right) &= \mathrm{E}\left(\prod_{k=i,j}\left(\theta_k\left(G_t - \mathrm{E}(G_t)\right) + \sigma_k\sqrt{G_t}W_t^{(k)}\right)\right) \\
&= \theta_i\theta_j \operatorname{Var}(G_t) + \theta_i\sigma_j \mathrm{E}\left(G_t\sqrt{G_t}\right)\underbrace{\mathrm{E}(W_t^{(i)})}_{=0} \\
&\quad +\theta_j\sigma_i \mathrm{E}\left(G_t\sqrt{G_t}\right)\underbrace{\mathrm{E}(W_t^{(j)})}_{=0} + \sigma_i\sigma_j \mathrm{E}(G_t) E\left(W_t^{(i)}W_t^{(j)}\right) \\
&= \theta_i\theta_j \operatorname{Var}(G_t) \\
&\quad +\sigma_i\sigma_j t \mathrm{E}\left(\left(\sqrt{b}F_t + \sqrt{1-b}U_t^{(i)}\right)\left(\sqrt{b}F_t + \sqrt{1-b}U_t^{(j)}\right)\right) \\
&= \theta_i\theta_j\nu t + \sigma_i\sigma_j t b E(F^2) \\
&= \left(\theta_i\theta_j\nu + \sigma_i\sigma_j b\right)\cdot t.
\end{aligned}$$

Proof of (9) in section 3.4 and (12) in section 4.1.

If

$$\begin{aligned}
X_t^{(i)} &= cM_t + \sqrt{1-c^2}Z_t^{(i)}, \\
X_t^{(j)} &= cM_t + \sqrt{1-c^2}Z_t^{(j)}
\end{aligned}$$

as in section 3.4 then

$$\begin{aligned}
\operatorname{Cov}\left(X_1^{(i)}, X_1^{(j)}\right) &= \operatorname{Cov}\left(cM_1 + \sqrt{1-c^2}Z_1^{(i)}, cM_1 + \sqrt{1-c^2}Z_1^{(j)}\right) \\
&= c^2 \operatorname{Var}\left(M_1\right) \\
&= c^2\left(\frac{\nu}{c^2}\left(c\theta\right)^2 + \left(\sigma\right)^2\right) \\
&= c^2\left(\nu\theta^2 + \sigma^2\right).
\end{aligned}$$

Since

$$\operatorname{Var}(X_1^{(i)}) = \left(\nu\theta^2 + \sigma^2\right)1,$$

we get

$$\operatorname{Cov}(X_1^{(i)}, X_1^{(j)}) = c^2.$$

The proof of (12) is identical.

Proof of (13) in section 4.1

This proof is identical to the one proposed by Vasicek [1987] for the Gaussian copula. Let N be the number of entities in the portfolio. Conditional on $M = m$, the probability of $0 \leq k \leq N$ defaults is then

$$\begin{aligned}\mathbb{P}_k(M=m) &= \binom{N}{k} \left(\mathbb{P}(X_i < C|M=m)\right)^k \left(1-\mathbb{P}(X_i < C|M=m)\right)^{N-k} \\ &= \binom{N}{k} \left(F_{Z_i}\left(\frac{C-mc}{\sqrt{1-c^2}}\right)\right)^k \left(1-F_{Z_i}\left(\frac{C-mc}{\sqrt{1-c^2}}\right)\right)^{N-k}.\end{aligned}$$

We suppress the index i of Z_i in the following, since all Z_i are identically distributed. The unconditional probability is therefore

$$\mathbb{P}_k = \binom{N}{k} \int_{-\infty}^{\infty} \left(F_Z\left(\frac{C-mc}{\sqrt{1-c^2}}\right)\right)^k \left(1-F_Z\left(\frac{C-mc}{\sqrt{1-c^2}}\right)\right)^{N-k} f_M(m)\mathrm{d}m.$$

If we substitute

$$s = F_Z\left(\frac{C-mc}{\sqrt{1-c^2}}\right),$$

we find for the percentage portfolio loss not exceeding x:

$$\begin{aligned}F_N(x) &= \sum_{k=0}^{[Nx]} \mathbb{P}_k \\ &= \sum_{k=0}^{[Nx]} \int_0^1 s^k(1-s)^{N-k}\mathrm{d}W(s)\end{aligned}$$

with

$$W(s) = F_M\left(\frac{1}{c}\left(\sqrt{1-c^2}F_Z^{-1}(s) - C\right)\right).$$

Since

$$\begin{aligned}\lim_{N\to\infty}\sum_{k=0}^{[Nx]}\binom{N}{k} s^k(1-s)^{N-k} &= 0 \quad \text{if } x < s \\ &= 1 \quad \text{if } x > s,\end{aligned}$$

the cumulative distribution function in the limit $N \to \infty$ is

$$\begin{aligned}F_{\text{portfolio loss}}(x) &= W(x) \\ &= F_M\left(\frac{1}{c}\left(\sqrt{1-c^2}F_Z^{-1}(x) - C\right)\right).\end{aligned}$$

References

[1] **Amato, Jeffrey D.** and **Gyntelberg, Jacob**. *CDS index tranches and the pricing of credit risk correlations.* BIS Quartely Review, March 2005.

[2] **Andersen, Leif** and **Sindenius, Jakob**. *Extensions to the Gaussian copula: Random Recovery and Random Factor Loadings.* Journal of Credit Risk 1 (1), 2004.

[3] **Bibby, Bo Martin** and **Sørensen, Michael**. *Hyperbolic processes in Finance.* in: Handbook of Heavy Tailed Distributions in Finance, Elsevier, pp. 211-248.

[4] **Burtschell, Xavier**, **Gregory, Jon** and **Laurent, Jean-Paul**. *A comparative analysis of CDO pricing models.* Working paper, BNP Paribas, April 2005.

[5] **Cariboni, Jessica** and **Schoutens, Wim**. *Pricing Credit Default Swaps under Lévy Models.* UCS Report 2004-07, K.U. Leuven, 2004.

[6] **Das, Sanjiv R.**, **Freed, Laurence**, **Geng, Gary**, **Kapadia, Nikunj**. *Correlated Default Risk.* Working paper, Santa Clara University, 2004.

[7] **De Servigny, Arnaud** and **Renault, Olivier**. *Default Correlation: Empirical Evidence.* Working paper, Standard and Poors, 2002.

[8] **Gibson, Michael S.**. *Understanding the Risk of Synthetic CDOs.* Finance and Economics Discussion Series, Federal Reserve Board, 2004.

[9] **Giesecke, Kay** and **Tomecek, Pascal**. *Dependent Events and Changes of Time.* Working paper, Cornell University, 2005.

[10] **Hull, John**, **Predescu, Mirela** and **White, Alan**. *The Valuation of Correlation-Dependent Credit Derivatives Using A Structural Model.* Working paper, University of Toronto, March 2005.

[11] **Hull, John** and **White, Alan**. *Valuation of nth to Default CDS Without Monte Carlo Simulation.* Journal of Derivatives, 12, 2, Winter 2004. pp. 8-23.

[12] **Hull, John** and **White, Alan**. *The Perfect Copula.* Working paper, University of Toronto, July 2005.

[13] **Joshi, Mark S.** and **Stacey, Alan M.**. *Intensity Gamma: A New Approach To Pricing Portfolio Credit Derivatives*. Working paper, available at www.defaultrisk.com, May 2005.

[14] **Kalemanova, Anna**, **Schmid, Bernd** and **Werner, Ralf**. *The Normal inverse Gaussian distribution for synthetic CDO pricing*. Working paper, available at www.defaultrisk.com, August 2005.

[15] **Luciano, Elisa** and **Schoutens, Wim**. *A Multivariate Jump-Driven Financial Asset Model*. UCS Report 2005-02, K.U. Leuven, 2005.

[16] **Madan, Dilip B.**, **Carr, Peter P.** and **Chang, Eric C.**. *The Variance Gamma process and Option Pricing*. European Finance Review 2, 79-105, 1998.

[17] **Sato, Ken-Iti**. *Lévy processes and Infinitely Divisible Distributions*, Cambridge Universiy Press, Cambridge, UK, 1999.

[18] **Vasicek, Oldrich**. *Probability of Loss on Loan Portfolio*. Memo, KMV Corporation, available at www.moodyskmv.com, 1987.